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THE

PHYSICAL REVIEW

A MORE ACCURATE AND MORE EXTENDED COSMIC-RAY IONIZATION-DEPTH CURVE, AND THE PRESENT EVIDENCE FOR ATOM-BUILDING

By Robert A. Millikan and G. Harvey Cameron
California Institute

(Received December 9, 1930)

Abstract

The cosmic-ray *ionization-depth curve* has been extended at both its upper and lower ends and made more accurate throughout. The *absorption coefficients* obtained directly from the slope of the curve run from $\mu=0.35$ per m. of water at the top (Pike's Peak) to $\mu=0.028$ at the bottom (80 m. or 262 ft. of water below the top of the atmosphere, thus bringing to light both softer and harder components than the authors had before found. Strong quantitative evidence is presented, on the basis of the Klein-Nishina formula, that the strongest and most absorbable cosmic-ray band arises from the act of formation of helium out of hydrogen. Striking qualitative evidence is found that the three more penetrating bands are due to the formation out of hydrogen of the only other abundant elements oxygen (C, N, O) silicon (Mg, Al, Si, S) and iron (Iron group). Two independent proofs are given that the cosmic-rays enter the earth's atmosphere as photons, namely, (1) they are quite uninfluenced by the earth's magnetic field, and (2) the ionization produced by them in a closed vessel does not increase continually in going to the top of the atmosphere but passes through a maximum. It is shown to follow that the cosmic rays, in coming from their place of origin to the earth have not passed through an amount of matter that is appreciable in comparison with the thickness of the earth's atmosphere and that they must therefore originate in interstellar space rather than in the atmospheres of the stars. Some participation of the nucleus in the absorption of cosmic-rays is brought to light.

1. Objectives

THE new series of measurements presented herewith on the relation between cosmic-ray ionization and depth in equivalent meters of water beneath the surface of the atmosphere was undertaken for two very specific reasons.

First.—Our preceding experiments, published in full in 1928,[1] had brought to light what seemed to us very striking evidence that the cosmic rays have their origin in the acts of formation "in the depths of space" of the atoms of the celestially common elements helium, oxygen (C, N, O), and silicon (Na, Mg, Al, Si, S) out of hydrogen. This evidence consisted:

[1] Millikan and Cameron, Phys. Rev. **32**, 533 (1928).

(a) In our experimental proof that within the limits of our observational uncertainty these rays show a uniformity of distribution, i.e., an independence of both latitude and of sidereal time. Both of these conclusions had been established by our trip to the Bolivian High Andes[2] in 1926, the first point not having been previously tested at all by other observers, the second having been so tested but with opposite results, though subsequent more careful experimenting by Hoffmann and Lindholm,[3] Steinke,[4] Hess,[5] and by one of us[6] has confirmed our conclusion;

(b) In our proof shown unambiguously by the curve itself, of the *banded character* of the rays;

(c) In the general rough agreement between our observed sequence of band-absorption coefficients and intensities, and the sequence of energies released, in accordance with Einstein's equation and Aston's curve, when the celestially most abundant elements helium, oxygen, and silicon are formed out of hydrogen; and

(d) In the rough agreement between the observed absorption coefficients and those computed from Dirac's formula connecting ray-energy with ray-absorption.

If our interpretation of our cosmic-ray results is correct, rays of still higher penetrating power should exist corresponding to the formation of still heavier elements out of hydrogen. Iron, at least, is abundant enough so that our theory suggested that we might find a cosmic-ray band, or cosmic-ray bands, of higher penetrating power than any we had thus far definitely observed. Such rays could be brought to light only by working at still lower depths in snow-fed lakes with electroscopes still more sensitive than any we had thus far used, and this new series of experiments was started in the spring of 1928 in part to test this point.

Second.—Our 1927–8 ionization-depth curve[7] showed characteristics at its upper end, i.e., at the highest altitudes at which we had ionization-depth readings, which seemed to be a bit out of line with the theory. Thus, our analysis, by means of the Gold tables, of our curve into its "monochromatic" absorption coefficients yielded as the strongest and most absorbable component $\mu = 0.35$ per meter of water, while the Dirac formula gave for the coefficient of rays due to the formation of helium out of hydrogen $\mu = 0.30$. While this was of the right order of magnitude, indeed quite close in view of our accuracy, *the divergence was in the wrong direction*, for our computated value 0.30 had applied to a monochromatic beam, but the actual beam, even if it entered the atmosphere as monochromatic would, where observed, have secondary, tertiary, etc., components, due to Compton encounters with electrons, and all such encounters tend, *until the beam has got completely into equilibrium with its secondaries*, to push down the observed μ, i.e., the μ ob-

[2] Millikan and Cameron, Phys. Rev. **31,** 163 (1928).
[3] Lindholm, Gerlands Beitrage zur Geophysik **22,** 141 (1929).
[4] E. Steinke, Zeits, f. Physik **42,** 570 (1927) and **48,** 647 (1928).
[5] Hess and Mathias, Wien. Ber. **137,** 327 (1928).
[6] R. A. Millikan, Phys. Rev. **36,** 1595 (1930).
[7] Millikan and Cameron, Phys. Rev. **31,** 925 (1928).

tained from the observed slope. The theory, then, shows no way by which the *computed* absorption coefficient of the pure radiation due to the formation of helium out of hydrogen can be *less than* the observed coefficient. It rather requires that the computed μ be at least as large as, and if equilibrium has not yet been obtained at the observation point, appreciably larger than the observed μ. We thought that our observations at high altitudes showed some slight indications that our uppermost lake-readings were accidentally high and we therefore wished to repeat these observations with more sensitive instruments and under better conditions, and to extend them, if possible, to still higher altitudes, hoping that a crucial test of our theory might come out of such more accurate high-altitude readings. In a word, then, we undertook the present series of observations to extend and improve our observational data at both the upper and the lower ends of the ionization-depth curve.

2. Technique

Our mode of procedure was precisely the same as that used in obtaining the last ionization-depth curve published in 1928, save that in order both to bring out weak effects at greater depths under water, and to obtain increased precision in high altitude observing, we were obliged to increase still further the sensitivity of our electroscope. To do this we built a new spherical instrument of steel, wall-thickness 3 mm, internal capacity 1622 cc, and filled it to a pressure of 30 atmospheres. (See Fig. 1.) This procedure, according to direct observational data to be presented later, multiplied our electroscope sensitivity 13.82 fold over that obtained under a pressure of one atmosphere. It represented, too, a sensitivity more than double that used in our published 1928 observations.[8] We determined the capacity of this electroscope with much precision by the method heretofore described,[8] finding it to be 0.979 absolute electrostatic units. Although we have not reached with this instrument the extreme limit of possible electroscope sensitivity, since increases in volume, slight decreases in fiber-capacity and still higher pressures are possible, yet, for the purposes for which it was to be used, this instrument came close to the limit of possible efficiency as a high frequency radiation detector.

In order to test the insulating properties of our quartz supports, before filling with air under pressure we pumped the chamber entirely free of air and found that it then leaked, to cite one particular test, from 226.5 volts to 226.0 volts *in four hours*. The support-leak was thus not as much as one-tenth of the slowest rate of discharge ever observed in this work, and a small fraction of a percent of the average rate. Further, it disappears entirely in our computations on rate of ion-formation because it is included in what we call the zero of our electroscope. Such minuteness of the leak of the supports, however, shows the completeness with which this source of error has been pushed into the background in these experiments.

As in all of our cosmic-ray work we followed here the procedure of reducing the fiber-deflection to volts at the instant at which it is read. This is done

[8] Millikan and Cameron, Phys. Rev. **31**, 922—5 (1928).

by calibrating the scale in the eyepiece just before or after each reading. This procedure eliminates completely all temperature effects on the fibers or on any of the electroscope parts. Thus in under-water work our procedure is to calibrate the scale for the deflection to be used under quiet reading-conditions *at the surface*, then to sink the electroscope to the desired depth and leave it for say twelve hours, then to bring it quickly to the surface, read and calibrate again. A small correction is applied for the discharge during lowering and raising by going through the operation of lowering and at once raising and seeing whether an observable change is detectable, or by doing so repeatedly for the sake of magnification and then computing the correction for a single operation. These corrections are small and quite accurately obtainable. In some of our earlier work we used self-registering electroscopes, but for this work we considered them of no advantage.

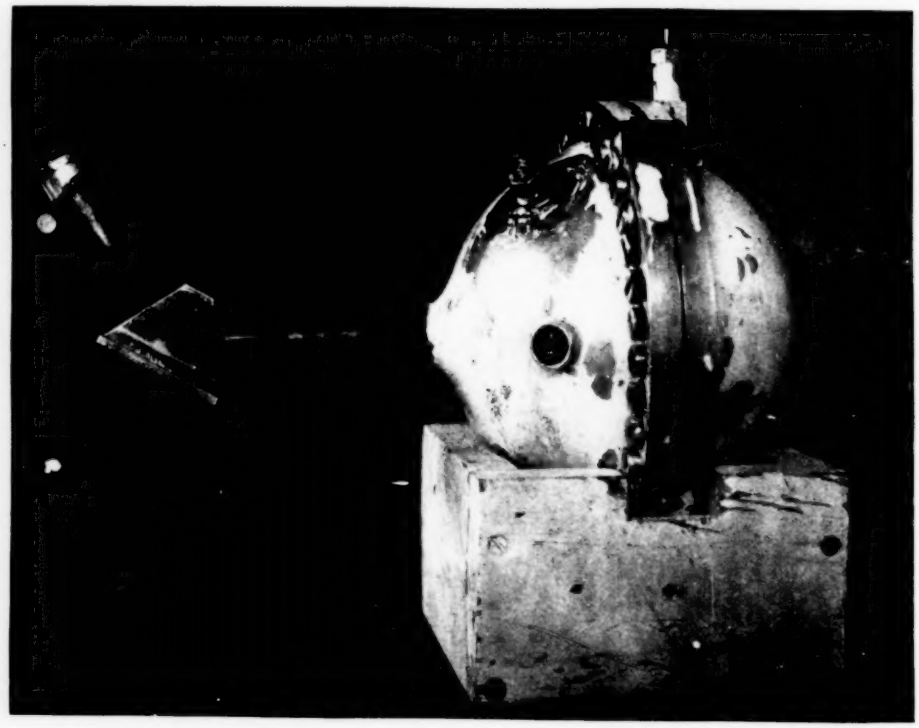

Fig. 1.

The electroscope with which all of the results herein reported were obtained is shown in Fig. 1. It was pumped up to a pressure of 30.14 atmospheres, one atmosphere being reckoned from the conditions existing in the Norman Bridge Laboratory at the time of filling, namely, 24°C, 74 cm pressure, and this pressure has held with no trace of leak for now two years since the time of filling.

The "electroscope constant" of this instrument in ions per cc per sec. is

$$I \text{ cc/sec} = \frac{\text{volts}}{\text{hours}} \times \frac{0.979}{300} \times \frac{10^{-10}}{4.770 \times 1622 \times 3600} = 1.172 \frac{\text{volts}}{\text{hours}}.$$

For observations on land this electroscope was provided with an accurately fitted, spherical lead shield 7.64 cm thick consisting of four layers of carefully shaped and fitted hemispherical shells, and two layers of shells divided into quadrants so as to fit closely the exterior electroscope wall, with its flange and bolts, and to facilitate the entrance of the necessary leading-in wires. The "water equivalent" of this lead shield as obtained by multiplying thickness by density was 85 cm. On account of the space occupied by the steel flange about which the inner layers of lead were fitted, the water equivalent was actually a trifle less than 85 cm, but in any case by careful experiments with radium and thorium in about the proportions in which they occur in surface rocks the percentage of local rays getting through this lead shield was found to be 2.4 percent, and save for this small correction-factor, always determined and applied to all readings, the rays inside the lead were pure cosmic rays and could be compared, as shown below, with readings at the same equivalent depth in water beneath the top of the atmosphere.

3. Readings in Snow-fed Lakes

The results reported in this section were taken partially during the summer of 1928 and partially during the summer of 1929 in the two different California lakes 250 miles apart, Arrowhead Lake (altitude 5100 feet) and Gem Lake (altitude 9120 feet). Since we showed in our first work in snow-fed lakes in 1925 that the measured intensity of the cosmic rays is a single valued function of the depth of the superincumbant atmosphere, it of course follows that it depends upon the barometric pressure, as has since that time been many times noted. Accordingly, in Table I the barometer height is given for each observation, or group of observations. At the higher levels, i.e., down to about sixteen meters beneath the top of the atmosphere, where readings at the same depth beneath the surface of the lake are made at different barometric pressures, the results are reduced to a common pressure by applying a small correction, which is computed from the Gold tables,[9] for the value of the absorption coefficient shown by the ionization-depth curve at the elevation considered. This computation, however, checks nicely with the observed slope of the curve and this slope is of course independent of any theory, so that these small barometric corrections may properly be said to be purely empirical, and hence free from any other than observational uncertainties. This correction is negligible below sixteen meters but amounts at Gem Lake to slightly more than 1% and at Pikes Peak to about 2% per tenth-inch of mercury. The under-water readings, taken in two lakes and over the period from August 1928 to September 1929, are all collected in Table I. It will be seen from columns 7 and 9 that these readings extend from a level equivalent to

[9] Gold, Proc. Roy. Soc. **A82,** 43 (1909).

8.25 meters of water beneath the top of the atmosphere down to a level of 80 meters, or 262.5 feet, beneath the top, and that in that range of levels, or depths, the observed rate of discharge of the electroscope changes by more than thirty fold, namely, from a value represented by an intensity of ionization within the electroscope of 64.1 ions per cc per sec. down to 2 ions per cc

TABLE I. *Ionization-depth readings in snow-fed lakes.*

Year	Date	Lake used	Depth in Water	Barometer Reading	Depth below Top of atmosphere	Ions per cc per second	Mean *I* cc/sec
1928	Sept. 11	Gem	.85 m	21.45	8.25	64.1	64.1
"	" 9	"	1.00 "	21.48	8.42	59.8	
"	" 9	"	1.00 "	21.48	8.42	60.4	60.1
"	" 11	"	1.00 "	21.48	8.42	60.2	
"	" 11	"	2.00 "	21.46	9.41	43.8	
"	" 11	"	2.00 "	21.41	9.40	43.8	43.8
1929	Sept. 6	Arrowhead	.82 "	24.81	9.40	43.8	43.8
"	" 7	"	1.50 "	24.77	10.06	37.5	37.5
"	" 6	"	2.00 "	24.82	10.58	33.5	33.5
1928	" 12	Gem	4.00 "	21.65	11.48	30.7	30.4
"	" 13	"	4.00 "	21.65	11.48	30.2	
1929	" 4	Arrowhead	3.00 "	24.94	11.59	29.6	29.5
"	" 5	"	3.00 "	24.94	11.59	29.4	
"	" 6	"	4.00 "	24.79	12.56	25.5	25.5
"	" 5	"	5.00 "	24.87	13.56	23.0	23.1
"	" 6	"	5.00 "	24.87	13.56	23.2	
"	" 7	"	6.00 "	24.74	14.55	21.1	21.1
1928	Aug. 22	"	6.25 "	24.92	14.86	20.2	20.6
"	" 23	"	6.25 "	24.93	14.86	20.8	
"	" 25	"	6.25 "	24.81	14.82	20.9	
"	" 23	"	8.25 "	24.92	16.85	17.10	17.33
"	" 24	"	8.25 "	24.89	16.84	17.57	
"	" 25	"	10.64 "	24.81	19.21	14.72	14.52
"	" 27	"	10.64 "	24.89	19.24	14.33	
"	" 25	"	15.90 "	24.81	24.47	10.22	10.23
"	" 26	"	15.90 "	24.83	24.49	10.24	
"	" 22	"	21.10 "	24.91	29.70	7.81	7.89
"	" 23	"	21.10 "	24.93	29.71	7.96	
"	" 20	"	26.25 "	24.81	34.82	6.07	6 .07
"	" 21	"	26.25 "	24.81	34.82	6.08	
"	" 19	"	30.35 "	24.79	38.91	5.34	5.21
"	" 18	"	30.35 "	24.80	38.91	5.08	
"	" 9	"	37.05 "	24.86	45.64	4.33	
"	" 10	"	37.05 "	24.86	45.64	4.09	4.25
"	" 11	"	37.05 "	24.86	45.64	4.34	
"	Sept. 9	Gem	43.00 "	21.48	50.42	3.68	3.62
"	" 10	"	43.00 "	21.39	50.41	3.56	
"	Aug. 17	Arrowhead	42.78 "	24.91	51.38	3.90	
"	" 15	"	42.78 "	24.91	51.38	3.88	3.79
"	" 12	"	42.78 "	24.89	51.38	3.60	
"	Sept. 8	Gem	50.00 "	21.42	57.40	3.30	3.30
"	" 12	"	60.00 "	21.36	67.38	2.49	2.49
"	" 10	"	72.5 "	21.39	79.90	1.95	2.00
"	" 11	"	72.6 "	21.45	80.00	2.05	

per sec. The absorption coefficients as computed from the Gold tables at successive points along this curve are given in Table II.

The curve starts a little higher up than does the 1928 curve, analyzed on page 927, Physical Review, Vol. 31, and it is significant that the absorption coefficient at the top is now a little higher than it was there, namely 0.27 in-

stead of 0.22, i.e., it is *larger, not smaller*, then before. Also, in keeping with this fact the bend, or knee, at about 10 meters is sharper than ever, as shown by the change from $\mu = 0.27$ to $\mu = 0.16$ between 9.5 m and 10.5 m, when on

TABLE II. *Absorption coefficients at various depths, in meters of water, below top of atmosphere.*

Depth in m. of water beneath top of atmos.	Absorption coef. μ	Depth in m. of water beneath top of atmos.	Absorption coef. μ
8.25– 9.5	0.27	20–30	0.045
9.5 –10.5	0.16	30–40	0.038
10.5–11.5	0.11	50–40	0.028
11.5–12.5	0.095	50–60	0.028
12.5–15.0	0.067	60–80	0.028
15.0–20.0	0.058		

the old curve we got $\mu = 0.20$ between 9.5 and 10.5. *Of course this means that the most absorbable cosmic-ray band springs into view from these figures more insistently than before.*

4. LAND-READINGS UP TO GREAT ALTITUDES

The highest altitude snow-fed lake used for the foregoing readings was Gem Lake (altitude 9120 feet) and for the sake of being free from the possibility of effects due to the radioactive emanations of the atmosphere (though over large bodies of water these effects are actually very small) we used no reading nearer the surface of Gem Lake than 0.85 m, a level corresponding to 8.25 m of water beneath the top of the atmosphere. At this level, as indicated above, the curve was already beginning to show departures in the wrong direction for satisfactory explanation from the standpoint of the Dirac formula. However, in accordance with the second of the objectives discussed in §1, the most significant data were to be expected at still higher altitudes, and in order to obtain accurate data at least a meter of water higher up we arranged for a series of land observations as follows:

With the aid of the radiations emitted by known quantities of radium and thorium set up at suitably chosen points, from 2 to 10 meters away, and all around our 7.64 cm lead screen we took readings when the screen was in place and when it was removed from our electroscope, and thus found that about 2.4% of the local radioactive rays from surface rocks and soils get through the lead screen and produce ionization within our electroscope. We then took a series of land observations in various localities, situated in widely different latitudes and at varying elevations from sea level up to 14,100 feet (the top of Pike's Peak), half a dozen or more readings being in general taken at each locality over a period of several days, first, when the lead screen was in place, then when the screen was removed. By comparing these observations with those taken at the same levels beneath the surfaces of snow-fed lakes the water equivalent of the lead screen was quite accurately determined, as shown below, and in this way the depth-ionization curve was reliably extended upward the equivalent of about a meter of water above the highest point obtainable in Gem Lake. This last meter proves to be of great significance for

TABLE III. *Ionization-depth readings at various elevations with and without lead screens.*

Date	Locality	Elevation	Latitude	Barometer readings	Depth below top of atmosphere	Mean ions outside lead	cc/sec inside lead	% cos. rays getting through lead	Mean ions corrected for local rays
Sept. 23–24, '28	Pike's Peak	14,100 ft.	39⅓	17.94 in.	6.195 m. water	288.7	94.2	61%	90.9
Sept. 9–10, '30	Pike's Peak	" "	"	17.84	6.161 " "	290.0	96.4	61%	93.2
Sept. 24, '28	Windy Point	12,200 ft.	"	19.23	6.640 " "	247.4	77.9	62%	75.0
Sept. 10, '30	" "	" "	"	19.29	6.663 " "	245.6			
Sept. 25–26, '28	Mt. Manitou	8,590 "	"	21.97	7.595 " "	188.9	53.6	64%	51.1
Sept. 21–22, '28	Colorado Springs	6,035 "	"	24.13	8.333 " "	113.5	43.51	67%	42.3
Aug. 19–20, '30	Lake Louise Canada	5,670 "	51½	24.41	8.432 " "	81.14	41.80	68%	41.3
Aug. 18, '28	Arrowhead Lk. (in boat)	5,100 "	34	24.89	8.596 " "	63.3	39.96	69%	39.8
Aug. 19, '28	Arrowhead Lk. (on shore)	" "	"	"	" " "	124.8	40.97	69%	39.3 } 39.5
Sept. 22–23, '28	Pueblo, Colo.	4,651 "	38½	25.47	8.796 " "	107.1	37.9	70%	36.6
July 26 to Aug. 4, '30	Pasadena	756 "	34	29.14	10.06 " "	90.7	29.56	76%	28.30
Aug. 25 to Sept. 1, '30	Churchill Manitoba	20 "	59	29.44	10.16 " "	58.87	28.48	76%	27.90

the purposes of the second objective (§1) and in general for the interpretation of the cosmic radiations. Table III contains the record of these land-readings, which were taken during the summer and fall of three consecutive years, 1928, 1929, and 1930. The two readings on Pikes Peak taken two years apart are rather noteworthy. They differ by 2%, but in view of the difference in the barometer reading this difference is not only in the right direction but of the right amount.

The figures given in the last and the third from the last columns are in all cases the means of from three to nine different readings, the fluctuations in which are illustrated, for example, by the nine consecutive readings taken on Mt. Manitou at about two hour intervals beginning at 10:30 a.m. These nine readings, in ions per cc per sec, run 54.7, 52.8, 55.5, 54.3, 52.5, 53.7, 52.4, 52.2, 54.3. Mean = 53.6.

5. The Depth-Ionization Curve and Its Significance

The graphical representation of the results in Tables I and III is given somewhat inadequately in Figs. 2 and 3,—inadequately because no small scale graph can reflect the consistency and precision of these readings. For this reason the readings themselves have been given in Tables I and III so that the reader may plot his own large-scale graph if he so desires. However, two important results stand out immediately and conspicuously from the graphs:

1. At great depths, i.e., between 40 m and 80 m (see Fig. 3) the readings are so consistent as to show that even a blanket of 80 m or 262 ft. of water is insufficient to absorb completely the cosmic rays. The curve has here reached a value of 2 ions per cc per sec., not a fiftieth of its value at Pike's Peak, *but it is still falling.* In order to find the zero of the instrument, or the ionization *when all external rays have become absorbed* and at the same time the absorption coefficient of these hardest rays, we analyzed the curve by the trial and error method between 40 m and 80 m with the aid of the Gold tables, and found that with a zero of 1.2 I per cc/sec. and an absorption coefficient of 0.028 per meter of water, *the whole long stretch of curve between 40 m and 80 m was very accurately reproduced. In a word, our curve shows at its lower end just such a band of hard rays as we had been looking for, and a single coefficient is adequate for the whole range between 40 m and 80 m.*

The significance of the value of this coefficient will be discussed presently, but the very existence of such a coefficient means that a hundred meters farther down, i.e., at 180 m, an ionization of about 0.03 I per cc/sec. should be observable by an instrument capable of detecting such an amount. We checked this conclusion qualitatively in 1928 by taking our electroscope and its lead shield down into a shaft 185 m deep, beside Lake Arrowhead, and finding, after allowing for the local rays, a fraction of an ion still left for the cosmic rays, though, on account of the location of the shaft *beside the lake* we could not reliably estimate the equivalent water depth.

However, Regener[10] has reported more dependable deep water observa-

[10] E. Regener, Die Naturwissenschaften **15**, 183 (1929).

tions which are in substantial agreement with the results here given. He reports his results in volts per hour, which is nearly the same as our ions per cc

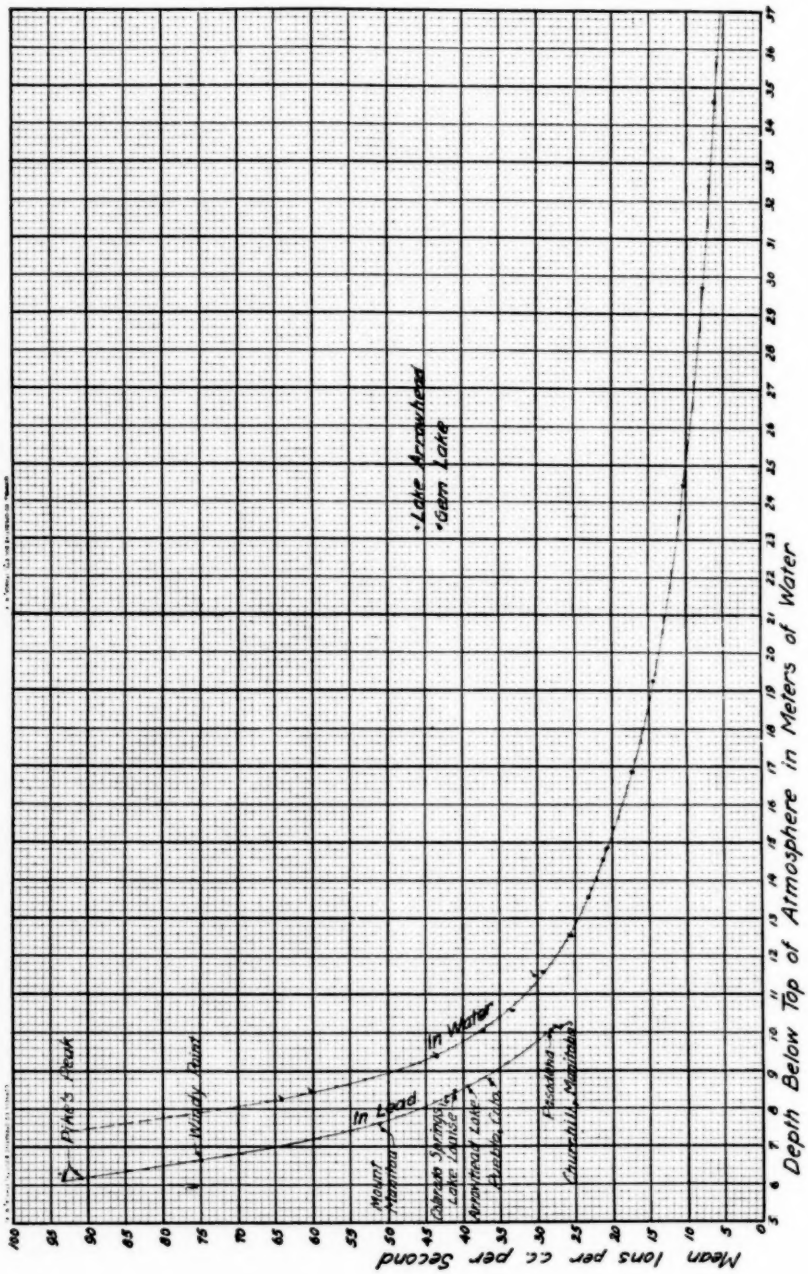

Fig. 2.

per sec., since the multiplying factor of 1.171 reduces in our case I volt/hour to I per cc/sec. His lowest really detectable reading is at 186 m, though he takes

one observation at 230 m. He obtains both his zero and his absorption coefficient by essentially the same trial and error method that we use, and we take

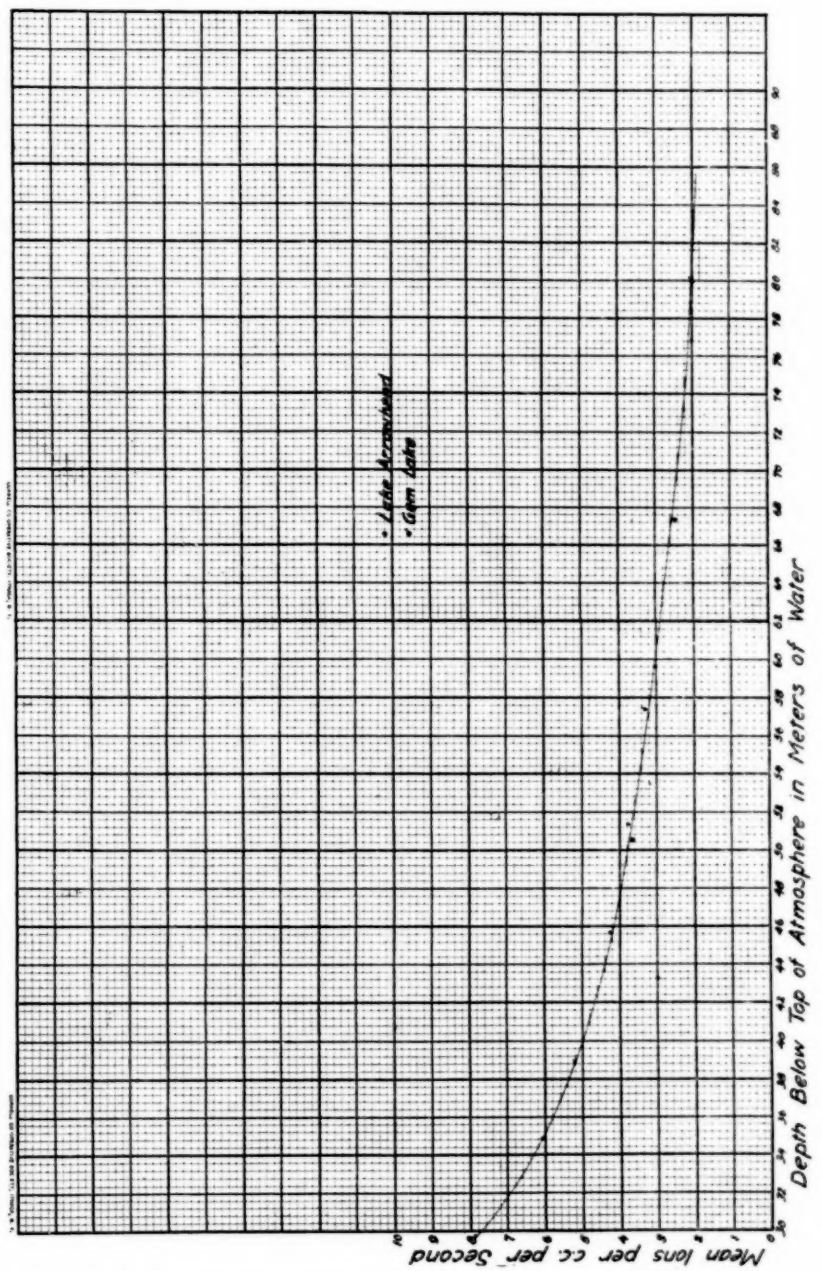

Fig. 3.

it that both he and we may have an uncertainty of as much as 0.2 I per cc/sec. in our zeros. Such a change in our zeros, if taken in the right direction for

both of us, would bring our coefficients into fairly good agreement. His value is 0.018 per meter of water. However, for reasons which we shall presently detail, the precise value of the coefficient for this most penetrating cosmic radiation is not particularly significant. The important result that appears both from Regener's deep water work and from our own is merely that there exists a component of the cosmic rays which can penetrate as much as 180 meters of water and which is therefore approximately twice as hard as the hardest component we had before *directly observed* on our former curve namely, $\mu = 0.05$ per m of water.

2. The second immediately striking result that appears from our curves is that the quite accurate observations we have now taken *at high altitudes*—up to 14,100 feet, show that the absorption coefficient now rises *far above anything the Dirac formula will in any way permit.* The absorption coefficient shown at the top of our 1928 curve was 0.22 per m of water, that at the top of our present curve taken in water is 0.27 per m of water, and that at the top of the land-curve is 0.35 per m of water. This new curve then brings out more strikingly than ever the banded structure of the cosmic rays.

We have analyzed this new curve with the aid of the Gold tables precisely as we did the former one in our 1928 article. We at that time found that the peculiar sharp bend in the curve at between 10 and 12 meters could not be reproduced without invoking *three definite bands* having roughly the relative frequencies 1, 4, 8. The new curve revealed the same necessity through the shape of its upper and middle portion, while its shape at great depths requires the introduction of a fourth band, as indicated above. Fearing that we might have become special pleaders for our banded structure, we asked Dr. Bowen, who had not thus far helped in this kind of analysis, to start from first principles with our curve and see, without any suggestions from us, what kind of structure it demanded. He proceeded without reference to any theory to build up with the aid of the Gold tables our observed curve out of four components—no smaller number would do—and in such a way that the synthetic and the observed curve fitted exceedingly nicely from one end to the other.

The components from which the synthetic curve was thus built up to yield the ionization *observed in our electroscope* were as follows:

TABLE IV. *Assumed absorption coefficients and intensities of synthetic curve.*

Assumed absorption coefficients	Assumed I_0 at top of atmosphere	Total $I = I_0/\mu$
0.03	33	1100
.10	80	800
.20	130	650
.80	141,000	176,250

The foregoing of course assumes that the ionization has its maximum value I_0 at the surface of the atmosphere (see below, where this is shown to be incorrect), and the last column then shows that the total energy of formation

of the iron group, the silicon group and the oxygen group is of the same general order of magnitude. This checks reasonably well with that we know of the relative abundance of these elements, and it would probably check better if suitable corrections could be made for the foregoing incorrect assumption. Table V then gives a comparison of the calculated and observed ionization.

TABLE V. *Comparison of synthetic and observed curves.*

Depth in meters	Calculated I	Observed I	Difference
7.5	89.7	90.8	−1.0
8.0	70.6	70.6	0
9.	48.2	48.1	− .1
10.	37.1	36.9	− .2
12.	26.4	26.5	+ .1
15.	19.1	19.1	.0
20.	12.53	12.55	+ .02
25.	8.87	8.75	− .12
30.	6.58	6.56	− .02
40.	3.93	3.83	− .10
50.	2.49	2.62	+ .13
60.	1.63	1.88	+ .25
70.	1.11	1.29	+ .18
80.	.74	.80	+ 0.6

Not only can the observed curve not be fitted accurately with less components than four, but also the four must be roughly of the foregoing type, though of course the lower bands can be split up into a finer structure if desired, i.e., 0.03 can easily be replaced by 0.02 and 0.04, for example. Not very much liberty, however, can be taken with the upper bands.

We shall now compare the results of this purely empirical study of our curve with the results computed by the Klein-Nishina formula,[11] the Einstein equation $mc^2 = E$, and Aston's curve. This formula has the form

$$\mu = \frac{2\pi N e^4}{m^2c^4}\left\{\frac{1+\alpha}{\alpha^2}\left[\frac{2(1+\alpha)}{1+2\alpha} - \frac{1}{\alpha}\log(1+2\alpha)\right] + \frac{1}{2\alpha}\log(1+2\alpha) - \frac{1+3\alpha}{(1+2\alpha)^2}\right\}.$$

Where $\alpha = h\nu/mc^2$, N = no. of electrons per cc. The values of α for the atom-building processes, according to Aston, are

$$II \rightarrow He = 0.029 \div 5.479 \times 10^{-4} = 52.9$$

$$II \rightarrow O = 0.1245 \div 5.479 \times 10^{-4} = 227$$

$$II \rightarrow Si = 0.232 \div 5.479 \times 10^{-4} = 423$$

$$II \rightarrow Fe = 0.48 \div 5.479 \times 10^{-4} = 876$$

5.479×10^{-4} is the atomic weight of an electron obtained from e/m spectroscopically determined. The numerical value of the constant factor is then

[11] Nature **122,** 399 (1928).

$$\frac{2\pi N e^4}{m^2c^4} = \frac{2\pi N e^2}{c^4}\left(\frac{e}{m}\right)^2$$

$$= \frac{2 \times \pi \times 6.064 \times 10^{23} \times 10}{(2.998)^4 \times 10^{40} \times 18} \times (4.77 \times 10^{-10})^2 \times (5.279 \times 10^{17})^2$$

$$= 0.16614.$$

Then Table VI is the comparison of μ obtained from the Klein-Nishina formula and those of the synthetically obtained curves. The closeness of the agreement at the top is most significant. The progressive departures as the atoms become heavier look at first sight like a difficulty, but the next section goes into possible causes of this behavior.

TABLE VI. *μ in meters of water.*

	Computed	Observed
H→He	0.7957	0.80
H→O	.2409	.20
H→Si	.1418	.10
H→Fe	.0754	.028

6. Conditions of Equilibrium between a Beam of Photons and its Secondaries Produced by the Compton Process

When a beam of photons strikes matter it is obvious that a certain thickness of matter must be traversed before the beam gets into equilibrium with its secondaries, tertiaries, etc., this condition of equilibrium being attained when as many of each kind of secondary is disappearing per second from the beam as is forming per second in it. While this process of getting into equilibrium is going on, the absorption coefficient of the beam, as measured by the rate of change with distance of the ionization produced per cc, is obviously smaller than it can be after equilibrium has been reached. Further, the absorption coefficient of the pure photon beam when it first strikes matter must be the same as that of the beam after it has got into equilibrium with its train of secondaries, provided the secondaries are more absorbable than the primaries,[12] for the reason that in this equilibrium condition the percentage of these secondaries is not changing at all as the beam moves on, the only element that is so changing being simply the number of primaries; but this is precisely the situation in which the beam found itself when it first entered matter.

[12] The apparent absorption coefficient when equilibrium is reached is equal to the absorption coefficient of the primary or the secondary, depending upon which has the smaller coefficient, the general relation being

$$I = \frac{I_0\mu}{\mu_2 - \mu_1}(e^{-\mu_1 x} - e^{-\mu_2 x}).$$

Of the last two terms that having the larger coefficient will die out, leaving the effective coefficient the one having the smaller value. It is practically certain from Bothe and Kolhörster's work that the coefficient of the beta-rays at sufficiently high frequencies approaches that of the photons, and it is entirely possible that it may even fall below it for the hardest rays.

One of us[13] has recently shown that the cosmic rays enter the earth's atmosphere as streams of pure photons. This means that the ionization in a closed vessel should not be a maximum at the top of the atmosphere, but that there must be an optimum position somewhere beneath the top where this maximum ionization is reached This is precisely what the 1922 high balloon flights of Millikan and Bowen, when taken in conjunction with Hess and Kolhörster's earlier and lower ballon flights proved experimentally to be the case. For in the 1922 flights a recording electroscope rose to a hight of 15.5 kilometers, at which height 0.92 of the earth's atmosphere had been left behind, and the total ionization recorded by the self-registering mechanism proved to be only about one-fourth of that calculated from the absorption coefficient of 0.57 per meter of water, which had been found by Hess and Kolhörster in rising in manned balloons to from 5 to 9 kilometers. The discrepancy seemed to be eliminated in 1923 when Kolhörster in experiments in the Alps got a coefficient only about 0.25 per meter of water, a value not in conflict with Millikan and Bowen's high flight. But up to the present day, though the lower value has been accepted, it has remained a mystery why the earlier European flights yielded such high values. This is now quite clearly explained, for the completely reliable curve shown in Fig. 1 is at the top rising quite as fast as did the Hess-Kolhörster curves, *while the 1922 work shows unambiguously that it cannot continue to do so up to 15.5 kilometers.* In other words, the coefficient has passed through a maximum somewhere between these two levels, and at 15.5 kms. has fallen back again to low values. This is obviously what, from the foregoing considerations, it must do if the cosmic rays enter the atmosphere as pure ether waves.

It will be seen from the foregoing that the Klein-Nishina formula, combined with Aston's measurements and Einstein's equation, yields quite accurately the observed absorption coefficient of the most absorbable band, when it is assumed that that band arises from the synthesis of helium out of hydrogen. Also there is good reason to assume that at the level corresponding to the top of our curve this radiation has already reached a condition of equilibrium with its secondaries, so that the comparison is probably here legitimate. The less absorbable the radiation, however, the farther must it traverse matter thus to get into equilibrium with its secondaries, and it is most illuminating to see how the absorption coefficients computed by the Klein-Nishina formula (see Table VI) for the formation of oxygen, silicon, and iron out of hydrogen *are progressively higher than the observed coefficients* as obtained from the curve, thus indicating that these progressively harder rays are farther and farther removed from the situation in which they have traversed enough matter to get completely into equilibrium with their secondaries. Further, such attainment of equilibrium should become more and more difficult the nearer the absorption coefficient of the beta-rays released by Compton encounters with photons approaches that of the photons themselves, and Bothe and Kolhörster's recent experiments[14] show that this condi-

[13] R. A. Millikan, Phys. Rev. **36,** 1595 (1930).

[14] Bothe and Kolhörster, Zeits. f. Physik **56,** 751 (1929).

tion is somewhat nearly approached for the harder rays, though it is probably not so for the softer components. It may be that for these very penetrating radiations the secondary electrons are more penetrating than the primary photons and consequently even when equilibrium is reached a lower absorption coefficient should be observed than that called for by the Klein-Nishina formula.[12]

7. Evidence for Atom Building

In a word, then, the general qualitative evidence that the cosmic rays are due to the formation out of hydrogen of the only four abundant groups of elements that there are, namely, helium, oxygen (C, N, O), silicon (Mg, Al, Si) and iron (the iron group), which elements, barring hydrogen, constitute more than 99 percent of all matter;[15] is quite extraordinarily good, but this evidence only becomes *quantitative* in the case of helium. That this band, however, which contains within itself probably more than 90 percent of all the cosmic-ray energy, has so closely the absorption coefficient predicted for it by the Klein-Nishina formula, taken in conjunction with Einstein's equation and Aston's curve, is exceedingly significant. By a process of exclusion we are well-nigh forced to adopt the synthesis of helium out of hydrogen as the origin of this cosmic-ray band; for the Einstein equation and Aston's curve leave us no other alternative, provided the Klein-Nishina formula yields a result of even the right order of magnitude for the relation between absorption coefficient and photon frequency or energy. The only other act that has been suggested, namely, the falling together of a positive and negative electron, *actually releases an energy 35 times that observed as obtained through the Klein-Nishina formula.* This formula has been proved by Millikan and Bowen,[16] by Chao,[17] by Tarrant[18] and by Meitner[19] to be approximately correct for the gamma rays from Th C″ while the cosmic ray band under consideration, in accordance with our direct measurement, *is but five times* as penetrating as these gamma rays, so that wholly apart from its theoretical credentials, the extrapolation from Th C″ up to the least penetrating cosmic-ray band is not a very long one. In other words, the Klein-Nishina formula ought to hold reasonably well for this softest cosmic-ray band.

8. Participation of the Nucleus in Cosmic-Ray Absorption

The Klein-Nishina formula, however, cannot be rigorously correct, for it makes the absorption proportional to the number of extra-nuclear electrons. We reported at the fall meeting of the National Academy in 1928 our definite evidence that the nucleus plays a role in cosmic-ray absorption. This evidence is found in the two curves of Fig. 2. If the mass absorption law held, the water equivalent of our lead screen would be 0.85 cm, this being obtained

[15] H. N. Russell, Astro. Phys. Jr. **70,** 11 (1929). The enormous abundance of H and He is the most striking feature of this article.

[16] Millikan and Bowen, Proc. Nat. Acad. **16,** 421 (1930).

[17] Chao, Proc. Nat. Acad. Sci. **16,** 426 (1930) and Phys. Rev. **36,** 1519 (1930).

[18] Tarrant, Proc. Roy. Soc. **129,** 342 (1930).

[19] Meitner, Naturwissenschaften **18,** 534 (1930).

merely from the thickness of the screen (7.64 cm) and the relative densities of water and lead. The actual water equivalent of the lead screen for the softer cosmic rays is seen from the distance apart, on the x-axis, of the upper parts of the two curves, to be 122 cm. This distance would of course be expected to increase for harder rays, and the curves show that this is indeed the case, since for rays of the hardness that are found at sea level (10 m) the water equivalent of the lead has become 170 cm. Chao[17] has recently brought to light a similar behavior for gamma rays, though of less magnitude, thus suggesting again the identity in nature of the gamma rays and the cosmic rays. The existence of such nuclear influences on absorption means that the Klein-Nishina formula must itself be but an approximation.

9. The Uniformity of Distribution of the Cosmic Rays

One of us has recently presented at length new evidence that the small fluctuations that have been reported in the intensity of the cosmic rays are due merely to changes in the thickness of the absorbing atmospheric blanket which surrounds the earth, and that the cosmic rays themselves are streaming into the earth with entire constancy and uniformity of distribution over the celestial sphere. The correctness of this conclusion with respect to latitude could scarcely be more beautifully attested than by the left-hand curve of Fig. 2. The observations there presented were taken with the same instrument under identical conditions as to observational technique, but at times extending over three different summers and in widely different altitudes and latitudes, the latter ranging from 34 to 59 degrees north. *Yet they fit with quite surprising exactness one single ionization-depth curve.* The observations at Lake Louise in Canada, latitude 51½, at Colorado Springs, latitude 39⅓, and at Lake Arrowhead, latitude 34, are especially comparable because taken at about the same heights, as are also the near sea-level observations at Pasadena (latitude 34) and Churchill, Manitoba (latitude 59).

This entire constancy in distribution of the cosmic rays is their most significant as well as most amazing property, and must mean, when taken in connection with their absorption coefficient, first, that the temperature existing even in the atmosphere of the sun, whence alone they could get to us from this star, and the same is true for other stars, is inimical to the union of hydrogen into the heavier elements, for hydrogen is present in enormous quantity in the sun's atmosphere. In the second place, these facts must mean that the cosmic rays do not originate in any places in the universe from which they are obliged to come to us *through any appreciable amount of matter whatever.* If they had done so they would on entering the earth be partly beta-rays and partly photons, and, on account of the earth's magnetic field, the beta-ray part would of necessity be stronger near the magnetic pole than at lower latitudes. *But no trace of such an influence can be discovered!* That, however, all about us "*in the depth of space*" hydrogen is somehow uniting into helium at least seems to us to be convincingly shown by the foregoing data, and that it is also uniting into the only other measurably abundant elements, the oxygen group, the silicon group, and the iron group, is strongly indicated, though

precise quantitative proof becomes here impossible because of the change—always a decrease—in the absorption coefficient of a beam of highly penetrating photons while it is getting into equilibrium with its Compton secondaries. This phenomenon—the existence of which was demonstrated by the 1922 Millikan and Bowen experiments—renders futile a more precise analysis of our curve than we have given above.

10. Summary

The foregoing results may be summarized as follows:

1. Much the most dependable ionization-depth curve which we have thus far presented, especially at very high and very low elevations, has been obtained and analyzed.

2. This curve shows, as did its predecessor, an unmistakable banded structure, and it further presents excellent quantitative evidence that the most intense and least penetrating band is due to the particular photon released when four hydrogen atoms unite to form an atom of helium.

3. Excellent evidence has been presented that these cosmic-ray bands enter the earth's atmosphere as ether waves and must penetrate far into it before getting into equilibrium with their secondaries; also, that until such equilibrium is attained the observed absorption coefficient is smaller than that of the initial monochromatic radiation. This means, as shown in the 1922 Millikan and Bowen experiments that there is a level beneath the top of the atmosphere at which the ionization due to a pure cosmic-ray beam is a maximum, this maximum moving rapidly farther down as the frequency of the initial photon increases.

4. The observed cosmic-ray curve has been shown to be consistent with the theory that it is made up of four bands due to the formation out of hydrogen of helium, oxygen, silicon, and iron the only atoms of sufficient abundance to render the radiation released by their formation detectable anyway, and it has been shown that the differences between the calculated and observed absorption coefficients of the photons produced by the formation out of hydrogen of oxygen, silicon, and iron are very nicely explained by the non-equilibrium theory given in (3), the departures all being in the right direction and increasing with frequency in the right way.

5. The constancy and the uniformity of distribution of the cosmic rays over the celestial sphere has been again brought strikingly to light, and the significance of this for the place and mode of origin of the cosmic rays has been again pointed out.

6. Some participation of the nucleus in the absorption of cosmic rays has been experimentally established.

We wish to express our appreciation to Professor Bowen for assisting, as indicated above, in the analysis of the curve.

RADIATION OF MULTIPOLES

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Abstract

The radiation of multipoles formed by putting together elementary dipoles is investigated according to the classical theory and wave mechanics. It turns out that the radiations are of two kinds:

The first part has the same frequency as the dipoles and is present only if the multipole structure is present even without vibration.

The second part is always present. If n, the order of the multipole, is even then the frequencies are 2ν, 4ν, up to $n\nu$. If n is odd then the frequencies are ν, 3ν, up to $n\nu$.

The relative intensities of subsequent multipoles are proportional to the square of the ratio: dipole amplitude to wave-length, as has been found in other cases. This calculation would apply to the infrared radiation of molecules.

RECENTLY there have appeared a number of papers on the radiation of higher poles;[1] this radiation seems to explain the presence of forbidden lines.[2] Although e.g. Rubinowicz' paper gives rather complete general formulas for the classical theory, it seems not useless to bring out the physical facts as clearly as possible by considering the case of a combination of linear oscillators, as they are present in molecules like CO_2 (two) or CCl_4 (four).

I. Classical Theory

We consider quadrupoles, but there is no difficulty in generalizing to higher poles. The simplest quadrupole consists of two equal dipoles vibrating in the direction of the line joining them but with opposite phases.

If the direction of vibration is the z-axis, the field of one dipole can best be described by a Hertz-vector Z

$$Z_x = Z_y = 0$$

$$Z_z = Z = a\frac{\cos 2\pi\nu(t - r/c)}{r} \tag{1}$$

$$E = \Delta Z - \frac{1}{c^2}\frac{\partial^2 Z}{\partial t^2}, \quad H = \frac{1}{c}\,\mathrm{rot}\,\frac{\partial Z}{\partial t}. \tag{2}$$

If we have now the two dipoles the distance l apart along the Z axis and vibrating with the same amplitude but opposite phase, we will get the re-

[1] I. Placinteanu, Zeits. f. Physik **39**, 276 (1926). A. Rubinowicz, Phys. Zeits. **29**, 817 (1928); Zeits. f. Physik **53**, 267 (1929); **61**, 338 (1930); I. Blaton, Zeits. f. Physik **61**, 263 (1930); I. Bartlett, Phys. Rev. **34**, 125 (1929); A. F. Stevenson, Proc. Roy. Soc. **A128**, 591 (1930); L. Huff and R. W. Houston, Phys. Rev. **36**, 842 (1930).

[2] I. Bowen, Proc. Nat. Ac. **14**, 30 (1928); Astrophys. J. **67**, 1 (1928); R. Frerichs and I. S. Campbell, Phys. Rev. **36**, 1460 (1930).

sultant Hertz vector at any point by taking the difference between the contributions from the two dipoles. The resultant Hertz vector will consist of three parts:

$$Z_2 = -\frac{a \cos 2\pi\nu(t - r/c)}{r} \frac{l}{r} \cos\theta. \tag{3}$$

This part will not contribute to the radiation in distant places on account of the factor l/r

$$Z_2 = 2\pi \frac{a \sin 2\pi\nu(t - r/c)}{r} \frac{l}{\lambda} \cos\theta. \tag{4}$$

This part does give radiation which will represent a spherical wave of the *same frequency* as the one which would be emitted by a dipole, although its intensity varies in a different manner with θ and its amplitude is diminished by the factor $2\pi l/\lambda$. The formula is

$$+ H_\phi = E_\theta = \frac{2\pi l}{\lambda} \cos\theta \left\{ \frac{4\pi^2}{\lambda^2} a \frac{\sin 2\pi\nu(t - r/c)}{r} \sin\theta \right\}. \tag{4a}$$

The total intensity emitted is the fraction $(8\pi^2/5)$ (l^2/λ^2) of that emitted by one dipole.

The third part is the most interesting one; it is always present, even if the two dipoles are not apart $(l=0)$ when at rest, that is to say if there were e.g. two negative charges vibrating on opposite sides of a positive charge with the same position of equilibrium. It arises from the difference of position caused by the vibration itself. That is to say, we have to take r now not as the distance from the point of observation to the dipole, or quadrupole, but to the vibrating charge. If we call the deviation from the equilibrium position in a given moment z', we have to put in (1)

$$r = r_0 + z' \cos\theta \tag{5}$$

and have for the dipole

$$\begin{aligned} Z &= a \frac{\cos 2\pi\nu(t - r_0/c) + 2\pi/\lambda z' \cos\theta \sin 2\pi\nu(t - r_0/c)}{r} \\ &= \frac{a \cos 2\pi\nu(t - r_0/c)}{r} + \frac{2\pi a^2}{\lambda} \cos\theta \frac{\sin 2\pi\nu(t - r/c) \cos 2\pi\nu(t - r/c)}{r} \\ &= \frac{a}{r} \frac{\cos 2\pi\nu(t - r_0/c)}{r} + \frac{2\pi a^2}{\lambda} \frac{\cos\theta}{2} \frac{\sin 2\pi 2\nu(t - r_0/c)}{r}. \end{aligned} \tag{6}$$

If the two dipoles of opposite phase are present, the first member drops out, and we have

$$Z_3 = \frac{2\pi}{\lambda} a^2 \frac{\cos\theta}{r} \frac{\sin 2\pi 2\nu(t - r_0/c)}{r}, \tag{7}$$

That represents a spherical wave, with the same angular distribution as Z_2, but with the *double* frequency and an amplitude proportional to the square of the amplitude of oscillation. The total amount radiated is $(4\pi^2/\lambda_0^2(a^2)2/5)$ times the amount radiated by one dipole.

Generalization to higher poles will give the following result: Assume that we have a pole whose order is n. n is defined in this manner: Let the unit vector $\boldsymbol{p}_i$ indicate the direction of one of the constituent unit dipoles which make up the higher pole. Let $\boldsymbol{r}$ be the unit vector in the direction of observation.

Then a pole of order n is one for which

$$\sum_i (\boldsymbol{p}_i \boldsymbol{r})^{n'} = 0$$

for all $\boldsymbol{r}$ and $n' < n$ and $\sum_i (\boldsymbol{p}_i \boldsymbol{r})^n \neq 0$ for some $\boldsymbol{r}$. In fact the left hand side is a spherical harmonic of order n. With this definition, a dipole has $n=1$, a quadrupole $n=2$, a tetrahedron $n=3$.

Let us now define a vector which will depend only on the orientation of $\boldsymbol{r}$ to a system of coordinates fixed in one pole

$$\boldsymbol{L}_n = \sum_i \boldsymbol{p}_i (\boldsymbol{p}_i \boldsymbol{r})^{n-1}$$

we will need later besides a scalar, defined by

$$\sigma_n = (\boldsymbol{L}_n \boldsymbol{r}')$$

where $\boldsymbol{r}'$ is a unit vector perpendicular to $\boldsymbol{r}$.

We find then two different kinds of radiation.

(1) If there is a "statical" multipole structure of order n and strength p_n present, we get a radiation (Hertz vector) of the same frequency as the frequency of the constituent dipoles, but with the factor

$$\frac{1}{n!}\left(\frac{2\pi}{\lambda}\right)^{n-1} p_n \boldsymbol{L}_n$$

(2) If the multipole has its origin only in the motion (that is if the field in equilibrium corresponds to a pole of higher order), part (1) is absent. But there is always present a radiation with a Hertz vector

$$Z = \frac{1}{n!}\frac{a^n}{r}\boldsymbol{L}_n \cos^{n-1} 2\pi\nu(t - r/c)\frac{\partial^{n-1}}{\partial r^{n-1}} \cos 2\pi\nu(t - r/c).$$

If n is odd, we have

$$Z = \frac{1}{n!}(-1)^{(n-1)/2}\frac{a^n}{r}\boldsymbol{L}_n\left(\frac{2\pi}{\lambda_0}\right)^{n-1} \cos^n 2\pi\nu(t - (r/c))$$

$$= \frac{(-1)^{(n-1)/2}}{2^{n-1}n!}\frac{a^n}{r}\left(\frac{2\pi}{\lambda_0}\right)^{n-1}\boldsymbol{L}_n\left\{\sum_{2s=0}^{2s=n-1}\binom{n}{s}\cos 2\pi(n-2s)\nu(t-(r/c))\right\} \quad (8)$$

If n is even

$$Z = \frac{(-1)^{n/2}}{2^{n-1}n!}\frac{a^n}{r}\left(\frac{2\pi}{\lambda_0}\right)^{n-1} L_n \left\{ \sum_0^{2s=n-2} \left(\binom{n}{s} - \binom{n}{s-1}\right) \sin 2\pi(n-2s)\nu(t-(r/c)) \right\}. \quad (8a)$$

Accordingly we get radiation which has the frequency $(n-2s)\nu$.

The only similar case where this seems to have been stated explicitly[3] is the case of n equal particles moving equidistant on a circle, which arrangement of course forms a pole of n fold multiplicity.

II. Wave Mechanics

In this case it is simpler to treat absorption instead of emission. The transition to emission is always easily made by using as external field the Landé-Dirac ghost field.

We assume a harmonic oscillator, whose Schrödinger equation is

$$\frac{d^2\psi}{dz^2} - \frac{8\pi^2 M}{h} 2\pi^2 M\nu^2 z^2 \psi = \frac{4\pi i M}{h}\frac{\partial\psi}{\partial t} \quad (9)$$

with the solution

$$\psi = \sum c_k \psi_k e^{-2\pi i k\nu t}\, \psi_k = e^{-\zeta^2/2} H_k(\zeta) b_n$$

with the abbreviation

$$\zeta = 2\pi\left(\frac{M\nu}{h}\right)^{1/2} z.$$

H_n is a Hermitian polynomial.

$$b_n = \left(2^{-k}\frac{1}{n!}\left(\frac{4\pi M\nu}{h}\right)^{1/2}\right)^{1/2}.$$

If there is now a perturbation function present of the form

$$V = e^{2\pi i s\nu t} v'$$

we find

$$\frac{dc_{k+s}}{dt} = \frac{2\pi i}{h} c_k \int V' \psi_k \psi_{k+s} d\zeta. \quad (10)$$

If we have a light wave polarized parallel to z and propagated parallel to x, its electric field will be

$$E_z = E\cos 2\pi\nu s(t - x/c). \quad (11)$$

The perturbation function is then given in the first case (of a statical multipole structure) by

$$V = \frac{z}{n!}\sigma_n p_n \frac{\partial^{n-1}}{\partial x^{n-1}} E_z$$

[3] J. J. Thomson, Phil. Mag. (VI) **6**, 673 (1903); G. A. Schott, Electromagnetic Radiation, Cambridge 1912, p. 102.

[4] See f. e. E. U. Condon and P. M. Morse, Quantum Mechanics, New York, 1930, p. 47. A. Sommerfeld, Wellenmechanischer Ergänzungsband, Braunschweig 1929 p. 17.

from which we get[5]

$$\frac{dc_{n+s}}{dt} = \frac{2\pi i}{h} c_k \left(\frac{2\pi}{\lambda_0}\right)^{n-1} p_n \sigma_n E \frac{1}{n!} \int z \psi_k \psi_{k+s} dz.$$

The latter integral gives of course the same selection rule as for a dipole $s = \pm 1$ and only modifies the intensity of the absorption by the same factors as in the corresponding classical example.

More interesting is the case arising from the existence of the oscillatory We will put the perturbation function multipole.

$$V = \frac{1}{n!} z^n \sigma_n e \frac{\partial^{n-1}}{\partial x^{n-1}} E_z$$

which leads to

$$\begin{aligned} \frac{dc_{k+s}}{dt} &= \frac{2\pi i}{h} c_k \frac{eE}{n!} \left(\frac{2\pi}{\lambda_0}\right)^{n-1} \sigma_n \left(\frac{1}{2\pi\left(\frac{M\nu}{h}\right)^{1/2}}\right)^{n+1} b_k b_{k+s} I_{k,s}{}^n \\ &= \frac{ic_k}{(\pi M \nu h)^{1/2}} eE \frac{\sigma_n}{n!} \left(\frac{h\nu}{Mc^2}\right)^{(n-1)/2} \frac{1}{2^{k+s/2}(k!(k+s)!)^{1/2}} I_{k,s}{}^n \end{aligned} \tag{12}$$

where

$$I_{k,s}{}^n = \frac{1}{b_k b_{k+s}} \int \zeta^n \psi_k \psi_{k+s} d\zeta = \int \zeta^n e^{-\zeta^2} H_k H_{k+s} d\zeta. \tag{13}$$

To evaluate this integral, we use first the following formula,[6] where an upper index j indicates j fold differentiation in respect to the argument

$$H_k{}^{(1)} = 2kH_{k-1}$$

and therefore

$$H_k{}^{(j)} = 2k \cdot (2k-2) \cdots (2k-2j+2) H_{k-j}. \tag{14}$$

If we write

$$H_k = \sum_i a_i{}^{(k)} \zeta^i$$

we have

$$H_k{}^{(j)}(0) = j! a_j{}^{(k)} = 2^j k(k-1) \cdots (k-j+1) H_{k-j}(0). \tag{15}$$

But on account of the formula[5]

$$H_k = (-1)^k e^{\zeta^2} \frac{d^k}{d\zeta^k} e^{-\zeta^2}$$

[5] Of course, we have been quantizing not the motion of one charge, but of a characteristic vibration (normal coordination). As these are independent (no natural interaction), there are only transitions between different states of the *same* vibration possible, as G. Dieke has kindly pointed out.

we have

$$H_{k-j}(0) = (-1)^{k-j}\left\{\frac{d^{k-j}}{d\zeta^{k-j}}\left(\sum_0^\infty(-1-)^l\frac{\zeta^{2l}}{l!}\right)\right\}_{\zeta=0}$$

$$= 0 \text{ for } k-j \text{ odd}$$

$$= (-1)^{(k-j)/2}\frac{(k-j)!}{\left(\frac{k-j}{2}\right)!} \text{ for } k-j \text{ even}$$

and finally

$$a_i^{(k)} = \frac{k!}{(i!)^2}2^i(-1)^{(k-i)/2}\frac{(k-i)!}{\left(\frac{k-i}{2}\right)!} \text{ for } k-i \text{ even} \qquad (16)$$

$$= 0 \text{ for } k-i \text{ odd}.$$

We now use Sommerfeld's method[7]

$$I = \int_{-\infty}^{\infty}\zeta^n\left(\frac{d^{k+s}}{d\zeta^{k+s}}e^{-\zeta^2}\right)H_k d\zeta$$

$$= \int_{-\infty}^{\infty}e^{-\zeta^2}\frac{d^{k+s}}{d\zeta^{k+s}}(\zeta^n H_k)d\zeta$$

$$= \int_{-\infty}^{\infty}e^{-\zeta^2}\frac{d^{k+s}}{d\zeta^{k+s}}\left(\sum_{i=0,1}^{k}a_i^{(k)}\zeta^{i+n}\right)d\zeta$$

$$= 0 \text{ for } k+s > k+n \text{ or } s > n$$

$$= \sum_i \int_{-\infty}^{\infty}a_i^{(k)}(i+n)(i+n-1)\cdots(i+n-k-s)\zeta^{i+n-k-s}e^{-\zeta^2}d\zeta \text{ for } s \leqq n$$

if we now put

$$i+n-k-s = l$$

we have

$$I = \sum_{l=0,1}^{n-s}a_{l+k+s-n}^{(k)}(l+k+s)\cdots l\int_{-\infty}^{\infty}e^{-\zeta^2}\zeta^l ds.$$

The integral is 0 if l is odd and $\pi^{1/2}\frac{(2l)!}{l!2^{2l}}$ if l is even $= 2j$.

Accordingly, we have

$$I_{k,s}^{(n)} = 0 \text{ if } s-n \text{ odd}$$

$$= \pi^{1/2}\sum_{j=0}^{(n-s)/2}\frac{(4j)!}{2^{4j}(2j)!}(2j+k+s)\cdots 2j$$

$$\cdot(-1)^{j+(s-n)/2}2^{2j+k+s+n}k!\frac{(n-s-2j)!}{\left(\frac{n-s}{2}-j\right)!}\frac{1}{(2j+k+s-n)!^2}$$

[6] Condon and Morse, reference 4, p. 50.

[7] Sommerfeld, reference 4, p. 59.

$$= \pi^{1/2}(-1)^{(s-n)/2}2^{k+s-n}\,k!\sum_{j=0}^{q}(-1)^j\frac{(4j)!}{(2j)!^2}\frac{1}{4^j}\;(2j+k+s)!\;\frac{(2q-2j)!}{(q-j)!}\;\frac{1}{(2j+k-q)!^2} \tag{17}$$

with $n-s=2q$.

That means that for n odd the transitions 1, 3, 5 $\cdots$ n and n even the transitions 2, 4, 6, $\cdots$ n are permitted, and accordingly the frequencies ν, 3ν, 5ν $\cdots$ $n\nu$ or 2ν, 4ν $\cdots$ $n\nu$ absorbed and emitted, in analogy to the classical theory. In normal cases, the formula are not as terrible as they look. For example, for $n=2$ (quadrupole) we have $s=2$

$$I_{k,2}^{(2)} = (\pi)^{1/2}2^k k!\frac{(k+2)!}{(k!)^2} = (\pi)^{1/2}2^k(k+1)(k+2).$$

If $n=3$ (tetraeder) there are possible two transitions
$s=1$, frequency ν

$$I_{k,1}^{(3)} = -(\pi)^{1/2}2^{k+2}k!\left\{\frac{(k+1)!2!}{(k-1)!^2} - \frac{4!}{(2!)^2}\frac{1}{4}\frac{(k+3)!}{(k+1)!^2}\right\}$$

$$= -(\pi)^{1/2}2^{k+2}\left\{2k^2(k+1) - \frac{1}{4}\frac{(k+2)(k+3)}{k+1}\right\}$$

$s=3$ frequency 3ν

$$I_{k,3}^{(3)} = (\pi)^{1/2}2^k\frac{k!(k+3)!}{k!^2} = (\pi)^{1/2}(k+1)(k+2)(k+3).$$

In general, we have for $s=n$, frequency $n\nu$

$$I_{kn}^{(n)} = (\pi)^{1/2}2^k\frac{k!(k+s)!}{k!^2} = (\pi)^{1/2}2^k(k+1)\cdots(k+2) \tag{18}$$

which will make the factor in (12) proportional to $[k\cdot(k+1)\cdots(k+n)]^{\frac{1}{2}}$ or $k^{n/2}$, if $k\gg n$; this corresponds to the fact that the amplitude of light emitted classically is proportional to a^n as $a^2\sim k\sim$ the energy of one dipole.

The relative transition probability from $k+s$ to k for a pole of the n order to the same probability of a pole of $(n-1)$ the order is

$$\frac{1}{n}\frac{\sigma_n^2}{\sigma_{n-1}^2}\left(\frac{I^{(n)}{}_{k,s}}{I^{(n-1)}_{k,s}}\right)^2\left(\frac{h\nu}{Mc^2}\right)^{1/2}$$

$h\nu/Mc^2$ is the ratio of the energy of vibration to the intrinsic energy and can be rewritten

$$\frac{M/2\overset{2}{v}_{\max}}{Mc^2} = \frac{1}{2}\left(\frac{v_{\max}}{c}\right)^2 = \frac{1}{2}\left(\frac{2\pi\nu a}{c}\right)^2 = 2\pi^2\left(\frac{a}{\lambda_0}\right)^2.$$

Rubinowicz[2] has found a similar result for atoms.

THE MOTION OF ELECTRONS IN ARGON

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(Received December 27, 1930)

Abstract

The motion of electrons in argon containing 1.53 percent N_2 has been investigated. For low fields the mobility is a constant, but rises rapidly as the field is increased due to the increase of the electron free path with increasing thermal velocity.

The mean free path of electrons in pure argon and in thermal equilibrium with the argon is 0.385 cm at 1 mm pressure.

IN A series of earlier papers[1] results of investigations on the motion of electrons in N_2, H_2, He and CO have been presented. The method used was the alternating potential method of measuring mobilities. Since in the case of the electron motion through a gas the mobility is in general not a constant, this method does not lead to a direct determination of the mobility. It is possible, however, as was done in this earlier work, to make an indirect comparison of the experimental results with the theoretically calculated mobilities. In all the cases presented so far, with the exception of N_2, a good agreement with the theory proposed by K. T. Compton[2] was shown to exist, provided it is assumed that the energy loss of an electron on impact with a molecule is greater than it would be if due to momentum transfer only. In N_2 the variation of the electronic free path with the electron velocity enters into the experimental results to such an extent that a check is impossible. Compton's theory takes no such variation into account.

In the present paper results on argon corresponding as much as possible to those on the other gases, are presented.

The argon used in these experiments contained, after being freed of oxygen, 1.53 percent nitrogen. This was not removed, but instead, as shall be seen, corrections were made for its presence. All the measurements were made at a pressure of 760 mm.

When the mobility is a constant, it may be calculated exactly from the voltage intercept of the mobility curves by the expression

$$K = \pi n d^2/(2)^{1/2}V. \tag{1}$$

If the mobility is a function of V as it will be for electrons, this equation gives an average value determined by the relation

$$\int_0^{1/2n} \frac{V}{d} K \sin 2\pi n t \, dt = \int_0^{1/2n} f(v) \frac{V}{d} \sin 2\pi n t \, dt.$$

Compton's theory gives

$$f(v) = \frac{a}{[1 + (1 + BV^2)^{1/2}]^{1/2}}. \tag{2}$$

[1] H. B. Wahlin, Phys. Rev. **23,** 169 (1924); **27,** 558 (1926); **35,** 1568 (1930).
[2] K. T. Compton, Phys. Rev. **22,** 333, (1923).

In Fig. 1 the average values of K calculated from Eq. (1) are plotted against the field strength (V/d) in which they were determined. For very low values of the field the mobility is a constant, and over this range the values calculated from Eq. (1) are nearly exact. In fields above 2 volts/cm the mobility rises sharply. The values found for this range serve merely to illustrate qualitatively the rapid increase in the mobility.

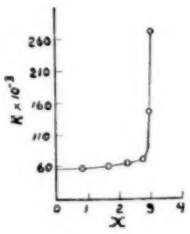

Fig. 1.

An examination of Eq. (2) shows that according to Compton's theory one should expect a decrease in the mobility with increasing voltage due to the increased thermal velocity of the electrons. In order to account for the discrepancy between theory and experiment it is necessary to assume a very rapid increase in the electron free path in argon with increasing thermal velocity. This is in agreement with the free path determinations of Townsend and Bailey[3] and others.

The results shown in Fig. 1 serve in one way as a check on Compton's theory. The equation giving the mobility as a function of the field strength (Eq. (2)) demands that for low field the mobility shall be a constant; i.e. that for low field the electrons shall be in thermal equilibrium with the gas.

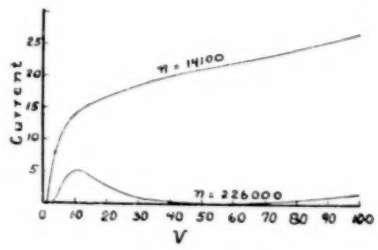

Fig. 2.

The fact that the mobility is nearly a constant as V approaches zero shows that apparently the thermal velocity does not vary sufficiently to produce any appreciable change in the free path.

In Fig. 2 two typical mobility curves are presented. The curve taken at a frequency of 14,100 cycles/sec is of the same character as those found for other gases. The one taken at the higher frequency is characteristic for argon. This frequency gives two values for the mobility. The variation in the shape of the mobility curve is due to the Ramsauer effect which in argon is much more pronounced than in other gases studied.

[3] J. S. Townsend and V. A. Bailey, Phil. Mag. **44**, 1033 (1922).

The limiting electron mobility in argon containing 1.53 percent N_2 is for low fields 59,600 cm/sec/volt/cm. The mean free path calculated from this value assuming

$$K = 0.75e\lambda/mc$$

is 0.363 cm at 1 mm pressure.

To correct for the presence of N_2 the following expression may be used

$$\frac{760}{\lambda_m} = \frac{P_n}{\lambda_n} + \frac{P_a}{\lambda_a}$$

where P_n is partial pressure of N_2, P_a the partial pressure of argon, λ_m the mean free path at 1 mm in mixture, λ_n the mean free path at 1 mm in N_2 and λ_a the mean free path at 1 mm in pure argon. This expression was used by Townsend and Bailey with a mixture of H_2 and A and was justified by them for this case. The corrected value for the mean free path when the electrons are in thermal equilibrium with the argon atoms is (assuming $\lambda_n = 0.0996$ cm) 0.385 cm at 1 mm pressure.

In Fig. 3 the author's value of λ (indicated by X) together with Townsend and Bailey's values are plotted as a function of the thermal velocity $\bar{c}$. This

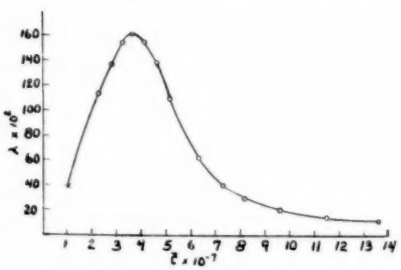

Fig. 3.

curve shows the prominence of the Ramsauer effect in argon, and also serves to explain the shape of the mobility curves obtained at high frequencies.

Summary

The results of the determinations of the mean free paths of electrons in gases, together with the values calculated from kinetic theory, are shown in Table I.

Table I.

	Obs.	KT	Obs./KT
H_2	0.041	0.0741	0.553
He	.066	.117	.565
CO	.069	.0391	1.75
N_2	.0996	.0393	2.53
A	.385	.0417	9.2

As is seen there is no systematic agreement between the observed and the kinetic theory values.

DISPERSION AND REFRACTIVE INDEX OF NITROGEN MEASURED AS FUNCTIONS OF PRESSURE BY DISPLACEMENT INTERFEROMETRY

By Clarence E. Bennett
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(Received December 15, 1930)

Abstract

The Lorentz-Lorenz relation has never been satisfactorily checked by experiment for pressures above atmospheric, because of a lack of trustworthy data on the refractive indices of pure gases as functions of pressure. Such data would also be particularly valuable in the study of polar molecules on account of Maxwell's law connecting the refractive index with the dielectric constant. Due, however, to the difference between optical and electrical frequencies the refractive index must be reduced to zero frequency before Maxwell's law can be applied. This necessitates a precise knowledge of dispersion which is not available for many pure gases. In the present paper a method of measuring simultaneously the dispersion and refractive index of a gas is discussed. The method is justified on theoretical and experimental grounds and applied in a practical way to the measurement of dispersion and refractive index of nitrogen over a range of pressures and at two different temperatures with a high degree of precision.

The results show that over a seven atmosphere range the Lorentz-Lorenz relation is followed and that the dispersion is a linear function of pressure. The value of the Cauchy constant B is found to be 1.95×10^{-16} per cm of mercury pressure. The index of refraction for nitrogen is measured for three wave-lengths at 0°C and 30°C with a limiting error in $(\mu-1)$ of one third of one percent. The dispersion information enables the values to be reduced to infinite wave-length with the result that a value for the dielectric constant is obtained which differs by less than two-thirds of one percent from that measured directly by Zahn. The values of μ under *NTP* conditions for the wave-lengths 4811A, 5893A, and 6362A are 1.0002991, 1.0002969, and 1.0002967 respectively.

Introduction

THE problem of measuring the index of refraction of gases has been attacked by many observers in the past, but precise results have been obtained with only a few substances. The most accurate determinations have been made for air, an admittedly complex substance, and in practically all cases the observations have been made at atmospheric pressure. The data on air obtained by Meggers and Peters[1] with a Fabry-Perot interferometer over the whole visible spectrum range, undoubtedly represent the best work of this sort ever attempted. Trustworthy data on the index of refraction of pure gases as a function of pressure, however, were scarce at the time the present research was started. Gale[2] in 1902 worked on air, about the constitution of which he knew very little, up to a few atmospheres with a Jamin interferometer, and Magri[3] in 1905 went up to fourteen atmospheres with air also using

[1] Meggers and Peters, Bulletin of the Bureau of Standards, No. 14, 1918–19, p. 697.
[2] H. G. Gale, Phys. Rev. **14,** 1 (1902).
[3] L. Magri, Phys. Zeits. **6,** 629 (1905).

a Jamin interferometer. Cheney,[4] using pure gases, in 1927 measured the refractive indices of nitrogen, ammonia, carbon dioxide, sulphur dioxide and air over a temperature range of a few hundred degrees at one atmosphere by means of a Fabry-Perot etalon, but so far as can be ascertained, no one with the exception of Phillips[5] who worked with carbon dioxide, has published any reliable data on pure gases at pressures higher than one atmosphere.

In the light of the Lorentz-Lorenz[6,7] equation such information should be particularly significant. Furthermore as a result of the work of Debye[8] and others in which dipole moments have been coordinated with dielectric constants, data on refractive index become especially valuable because of Maxwell's law connecting the dielectric constant with the refractive index. This necessitates a knowledge of the dispersion of the medium because dielectric constants are measured at very low frequencies compared with optical frequencies.

Some years ago Carl Barus[9] made a few preliminary observations on the index of refraction of air with his displacement interferometer, as a result of which, it appeared possible to measure both refractive index and dispersion simultaneously. This work was only qualitative and no attempt was made to secure high precision, yet the experiments showed that such precision might be expected if the instrument were properly developed. It was the particular good fortune of the writer to be able to assemble an instrument of this type with the personal assistance of Professor Barus, and to adapt it for this sort of work.

Displacement Interferometry

The optical arrangement of the displacement interferometer is shown in Fig. 1. When the system is properly aligned so that the mirror M' is made accurately parallel to the virtual image of the mirror M as seen through the half silvered mirror HS, the interfering light beams, if viewed through the telescope T on the front of which is mounted a ruled grating, display a beautiful interference pattern consisting of black concentric oval fringes superposed on a continuous spectrum. The optical path difference necessary to produce this result is a known function of the thickness, the index of refraction, and the dispersion of the glass half silvered mirror.

These fringes are very interesting. They have been studied at length by Barus,[9] and Birchby[10] has more recently described and explained fringes of a similar nature. Like Michelson fringes they expand or contract with every

[4] E. W. Cheney, Phys. Rev. **29,** 292 (1927).

[5] P. Phillips, Proc. Roy. Soc. London **97A,** 225 (1920).

[6] H. A. Lorentz, "Theory of Electrons."

[7] Meggers and Peters, reference 1, credit Magri, reference 3, with preferring $(\mu^2-1)/(\mu^2+1)/(\rho)=C$ but this is a mistake resulting from a typographical error in Magri's publication as anyone can see by checking the calculations.

[8] P. Debye, "Polar Molecules," 1929.

[9] Carl Barus, Carnegie Foundation of Washington Publications, No. 149, I, II, III; No. 229; No. 249, I, II; 1911 to 1917.

[10] W. N. Birchby, Proc. Nat. Acad. **10,**, 452 (1924) also **13,** 216 (1927).

motion of the micrometer screw, but continuous motion causes the whole system to move bodily along the spectrum from violet to red as the path from the micrometer mirror to the half silver is made shorter. Using for a half silvered mirror a plate of glass 0.25 inches thick, a motion of the micrometer mirror of 0.17 mm corresponds to a shift of the fringes from the red to the blue-green region of the spectrum. This means that movements of the fringe system along the spectrum corresponding to displacements of the mirror can be calibrated with a good micrometer. Hence the instrument is particularly well suited to measure displacements greater than can conveniently be measured with a Michelson interferometer, because it is unnecessary to count fringes, yet the accuracy is about the same.

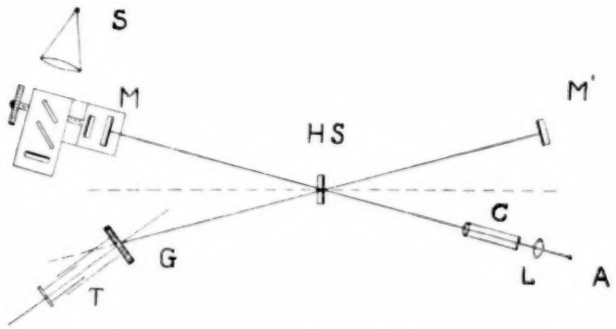

Fig. 1. Optical arrangement of a linear displacement interferometer.

The fringes are visible simultaneously in all spectrum orders and remain so even for very wide slit openings, the visibility however, being generally best for reduced illumination. When sunlight is used as a source, the dark Fraunhofer lines constitute fixed lines of reference by means of which measurements of the displacement of the fringe system can be made. Ordinarily with a carbon arc the yellow sodium line is bright enough to stand out against the continuous spectrum background and serve as a reference line.

By a geometrical consideration of the reflected and refracted rays Barus derived the fundamental relations of displacement interferometry. But let us now consider in quite an independent manner how the instrument can be used to measure the index of refraction of a gas. Suppose the interferometer to be so adjusted that the oval fringes are centered on a given spectrum line, for example the sodium D line, when a column of gas of length e is in one arm of the interferometer. Assume μ_1 to be the index of refraction of this gas, μ_A to be the index of refraction of the medium in which the micrometer mirror is displaced (air), and if there are any other substances in the path let their refractive indices be denoted by $\mu_2 \cdots \mu_n$.

With the gas in the path, the optical path difference from the half silver to the mirrors M and N is

$$n\lambda = \mu_1 e + F(\mu_2 \cdots \mu_n) - P \qquad (1)$$

where λ is the wave-length of the spectrum line on which the fringe system is centered, $\mu_1 e$ is the optical path length of gas, $F(\mu_2 \cdots \mu_n)$ is the path length of all remaining substances in the path, and P is the optical length of the opposite path. F and P are absolutely independent of the nature of the gas in question and are not affected by evacuating the tube.

When the gas is removed from the path (by evacuation of the containing tube) and the micrometer mirror is displaced to recenter the fringe system on the given spectrum line

$$n'\lambda = e + \mu_A \Delta x + F(\mu_2 \cdots \mu_n) - P \tag{2}$$

where μ for vacuum is of course one, and Δx is the air displacement of the micrometer mirror.

The radial distance between the central spot and the first fringe ring is observed to be greater than that between any two successive rings. Hence in accordance with Cornu's principle, the fraction of a fringe shift for the change of one wave-length must be a minimum at the center of the system, i.e.

$$\frac{\partial n'}{\partial \lambda} = \frac{\partial n}{\partial \lambda} = 0.$$

Thus the fringes near the center of the configuration are relatively coarse and far apart, whereas the outermost ones are very fine and close together.

Dividing Eqs. (1) and (2) by λ, differentiating each with respect to λ, setting the resulting equations equal to zero, subtracting one from the other, and solving, we get

$$\Delta x = \frac{e\left[(\mu_1 - 1) - \lambda \dfrac{\partial \mu_1}{\partial \lambda}\right]}{\mu_A - \lambda \dfrac{\partial \mu_A}{\partial \lambda}}. \tag{3}$$

The denominator for all practical purposes is one, whence

$$(\mu_1 - 1) = \frac{\Delta x}{e} + \lambda \frac{\partial \mu_1}{\partial \lambda}. \tag{4}$$

Dispersion

Eq. (4) shows that before the index of refraction of a gas can be determined by displacement interferometry some knowledge of its dispersion is necessary. Further considerations, however, show that the dispersion can be obtained directly with the instrument from the values of the displacement Δx taken for two different wave-lengths in the spectrum.

Consider the Cauchy dispersion relation

$$\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots.$$

The data of Meggers and Peters[1] show that for air, two arbitrary constants A and B are sufficient to represent the facts. The writer has taken these data for three different temperatures 0°, 15°, and 30°C and plotted $(\mu-1)$ aginst $1/\lambda^2$ on a very large scale (a graph 50×70 cm) as shown in a

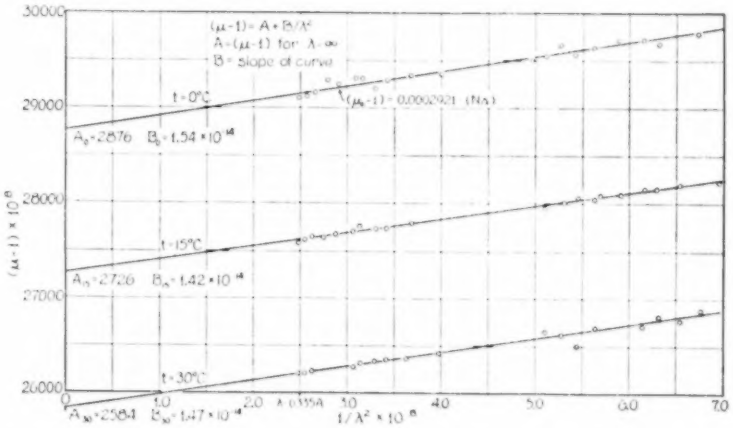

Fig. 2. Dispersion data for air plotted from observations of Meggers and Peters.

reduced photograph in Fig. 2. In each case the points lie rather well on straight lines, indicating that two arbitrary constants are sufficient to represent the data.

If now $(\mu-1)=A+B/\lambda^2$ whence

$$\frac{\partial\mu}{\partial\lambda}=-\frac{2B}{\lambda^3}. \tag{5}$$

Then Eq. (4) reduces to

$$(\mu_1-1)=\frac{\Delta x}{e}-\frac{2B}{\lambda^2}. \tag{6}$$

Now writing Eq. (5) twice for two different wave-lengths λ_1 and λ_2 and subtracting the one from the other

$$(\mu_1-\mu_2)=B\left(\frac{1}{\lambda_1^2}-\frac{1}{\lambda_2^2}\right). \tag{7}$$

For these two wave-lengths Eq. (6) gives

$$(\mu_1-1)=\frac{\Delta x_1}{e}-\frac{2B}{\lambda_1^2} \tag{8}$$

$$(\mu_2-1)=\frac{\Delta x_2}{e}-\frac{2B}{\lambda_2^2}. \tag{9}$$

Where B is assumed to be constant throughout the visible spectrum as the data of Meggers and Peters show to be true for air. Solving Eqs. (7), (8), (9)

$$B = \frac{(\Delta x_1 - \Delta x_2)}{3c\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right)}. \tag{10}$$

Now consider again Eq. (6) rewritten so as to read

$$\frac{\Delta x}{e} = (\mu - 1) + \frac{2B}{\lambda^2}. \tag{6a}$$

Experiments on nitrogen which will be described later, show conclusively that Δx is a linear function of pressure when values of Δx are plotted against pressure. Therefore, since e is a constant, $[(\mu-1)+2B/\lambda^2]$ must be a linear function of pressure, meaning that either each term separately varies directly with pressure or that there is some involved relation connecting the two terms in such a way that their sum varies directly with pressure if separately they do not. Since $\frac{2}{3}(\mu-1)$ is a first approximation to $(\mu^2-1)/(\mu^2+2)$ we should expect $(\mu-1)$ to be directly proportional to pressure by the Lorentz-Lorenz relation, and so B must, therefore, be a linear function of pressure. This means that B/p is a constant which can be determined directly by the difference in the slopes of the lines $\Delta x_1/p = \text{constant}$ and $\Delta x_2/p = \text{constant}$.

Dividing Eq. (10) by p it appears that

$$B_0 = \frac{B}{p} = \frac{\left(\frac{\Delta x_1}{p} - \frac{\Delta x_2}{p}\right)}{3c\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right)} \tag{11}$$

B_0 here means B per cm pressure. The B used in the previous work can now be interpreted to mean B at pressure p.

It follows then that the displacement interferometer can be used to determine the dispersion factor B of a gas, first finding B_0 from the difference in the slopes of the $(\Delta x, p)$ lines for two different wave-lengths and then multiplying by the pressure to get the value of B corresponding to that pressure. After B has been determined $(\mu-1)$ can be determined at the same pressure by Eq. (6). The value of $(\mu-1)$ can be reduced to its equivalent at 0° and normal pressure if it is not so measured, by assuming the Lorentz-Lorenz equation to be at least approximately true, whence

$$\frac{\mu - 1}{\rho} = \frac{\mu_0 - 1}{\rho_0} \tag{12}$$

$$\therefore \quad (\mu_0 - 1) = (\mu - 1)\frac{760}{p}\,\frac{T}{273.1}. \tag{13}$$

Preliminary Experiment

An interferometer with arms something over a meter long was assembled according to Fig. (1), and measurements of the refractive index of dry air

were made at atmospheric pressure using for the dispersion correction the value given by Fig. 2. The average of some twenty or more independent observations gave a resulting value for $(\mu_0 - 1)$ of 0.0002921 with a mean deviation of ± 0.0000007. This is exactly the value given by Meggers and Peters.[1] Hence the conclusion is drawn that the above theoretical considerations have been experimentally justified.

Description of the Apparatus

The arrangement of the apparatus is sketched in Fig. 1. A photograph of the actual set-up in the basement of the laboratory is reproduced in Fig. 3. The interferometer mirrors were made from plates of optical glass 3 inches square obtained from Bausch and Lomb. The micrometer mirror M was fastened to the movable table of a small Michelson interferometer which had a well-made micrometer screw about 3/16″ diameter of 0.5 mm pitch. The

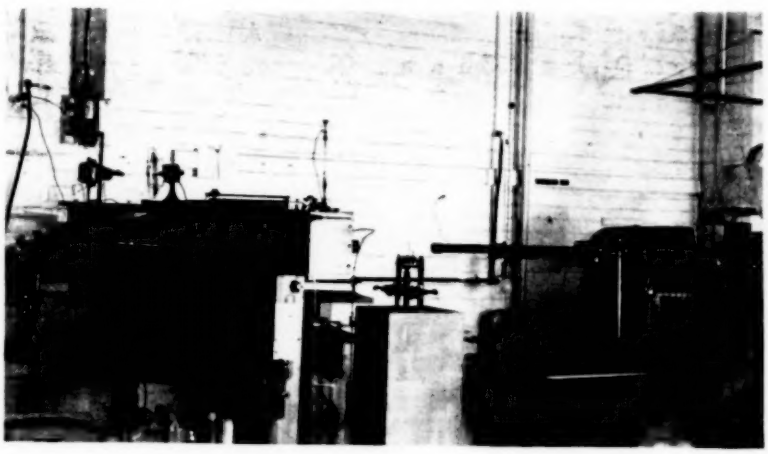

Fig. 3. Photograph of apparatus.

Michelson inteferometer was also set up so that it could be used as an auxiliary instrument as indicated in the figure, for the purpose of calibrating the screw over the range actually used, by counting Michelson fringes from an independent monochromatic source of light.

The Michelson interferometer and the two other mirrors of the displacement interferometer were mounted on independent cement piers which rested on the ground and were free from contact with the building. They extended up through the floor in the form of shafts one foot square and about three feet high.

The illuminating system consisted of an arc light, a lens and a collimator all mounted on a table. To produce a zinc spectrum for work on dispersion, the arc was loaded with pieces of zinc wire inserted into drilled carbons. The arrangement was such that a concave mirror could be substituted for the lens to make it possible to focus sunlight from a hole in the floor of the room above, upon the slit of the collimating telescope to produce a solar spectrum. The

oval fringes were observed with a 25 cm focal length telescope having a gelatin grating, with approximately 1150 lines to the inch, mounted normally in front of the objective. The arc light, the collimating telescope, the table upon which they rested, and the Michelson interferometer, were mounted upon squares of sponge rubber to minimize the effects of vibration. Even with the heavy cement piers completely separated from the building all trouble due to vibration was not completely eliminated, although with them the fringes occasionally steadied down for short intervals. Ordinarily in the day time the quiet intervals were neither as long nor as regularly spaced as in the evening, and for this reason most of the readings were taken in the evening.

The experimental refraction tube containing the gas to be experimented with, was supported in the working arm of the displacement interferometer. It was made of Shelby cold drawn steel tubing $\frac{1}{8}$ inch wall and approximately one inch bore. The ends were closed by disks of high quality optical glass, obtained from Bausch and Lomb, 1.15 inches in diameter and 0.25 inches thick, with accurately plane parallel surfaces. A second tube just like the first but containing air at room temperature and pressure was used in the

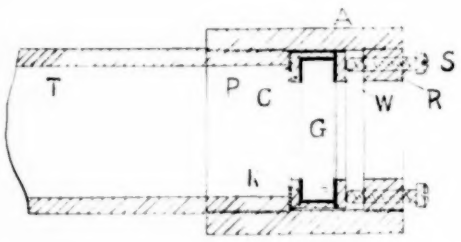

Fig. 4. Method of fastening glass ends to the experimental tube.

other arm of the interferometer to optically compensate for the experimental tube. The length of the refraction tube inside the end plates measured 100.30 cm.

The packing shown in detail by Fig. 4, was used to fasten the glass ends to the tube, and proved very satisfactory. The end of the tube T was carefully tinned and, while still hot, was screwed into a hot threaded sleeve A to form a very tight fit. C is a thin-walled steel cup carefully machined down to fit the glass disk G which is cemented into it with a thin layer K of de-Khotinsky cement. The steel washer W is also cemented to the glass, and the rim of C projects out beyond G just enough to take up the strain when W is pressed against it. P is a very thin sheet of rubber (dental dam) also cemented to C. This whole unit P, C, K, G, W, was set in place and the ring R was screwed in about flush with A. Then by means of eight equally spaced screws S the unit was pressed tightly up against the end of the tube T.

The experimental tube was mounted in a water bath, by means of which the temperature was kept uniform and constant to 0.01° c.

A small McLeod vacuum gauge was used for the purpose of registering pressures less than 0.1 mm mercury obtained with a Cenco Hyvac pump.

Pressures above atmospheric were measured with a mercury column. The lower end of this column was a length of 3/16″ steel tubing welded through a hole in the plug of a commercial iron mercury flask, and just long enough nearly to reach the bottom of the container. The upper end was a glass tube fastened to a brass centimeter scale, some 350 cm long in two sections, mounted on the side of a wooden ladder just long enough to reach from the floor to the ceiling beams, and fastened rigidly at each end.

The pressure line connecting the various pieces of apparatus; the gas source, the mercury column, a dial gauge (which was included for rough work), and the experimental refraction tube was made up of ⅛″ steel tubing. Connections were made by means of soldered sleeves or carefully built steel unions. Each piece of apparatus could be isolated from the rest by special high pressure steel stopcocks.[11]

Experimental Procedure and Observations with Nitrogen

Water pumped nitrogen was obtained in commercial containers under about 2000 lbs. pressure, guaranteed pure except for 0.5% water and 0.5% oxygen. The oxygen was removed by passing the nitrogen through a steel tube full of sheet copper clippings maintained at a temperature of 500–600°C into an auxiliary steel tank where it was stored under a pressure of 600 lbs. The water vapor was removed by phosphorus pentoxide contained in three tin cans suspended by copper wires inside this auxiliary tank.

The nitrogen thus stored was allowed to stand a few days to become dry. The system was thoroughly rinsed out with it and observations were carried out in the following manner: A vacuum of 10^{-2} mm of mercury was obtained, and after a reading of the barometer had been taken, the temperature of the bath being kept constant at 30°C, the fringes were very carefully centered on the blue (4811A) zinc spectrum line. The reading of the micrometer was recorded, and then the screw was so turned as to cause the fringes to proceed through the spectrum to the yellow (5893A) sodium line. This process was continued to bring the fringes up to the orange (6362A) zinc line. Then a small quantity of nitrogen was admitted to the system, and after the temperature and pressure had come to equilibrium, the temperature of the room being usually several degrees below 30°, the reading of the mercury column was noted, and readings of the micrometer were made with the fringes centered upon each of the three spectrum lines. This process was continued until readings had been made at intervals of approximately one atmosphere up to the limit of the gauge, i.e. a pressure of about 475 cm of mercury above vacuum. At this pressure due allowance was made for the back lash of the screw in the opposite direction, and readings were made for settings of fringes at each spectrum line, this time in the order of orange, yellow, and blue, for decreasing pressures at intervals of approximately one atmosphere, ending with a reading for vacuum.

After several runs of this sort a very careful run was made at a temperature of 0°C obtained by packing the constant temperature bath with ice.

[11] F. G. Keyes and Jane Dewey, Jour. Opt. Soc. Am., **14**, 491 (1927).

The observations obtained in these runs are recorded in Tables I and II.

TABLE I. *Data on nitrogen at 30°C.*

No.	Pressure (cm Hg)	Δx_1(cm)4811A (obs.)	(calc.)	Δx_2(cm)5893A (obs.)	(calc.)	Δx_3(cm)6362A (obs.)	(calc.)
1	148.89	0.05540	0.05573	0.05398	0.05439	0.05402	0.05412
2	150.72	.05628	.05641	.05495	.05506	.05495	.05479
3	152.27	.05605	.05699	.05588	.05562	.05538	.05535
4	153.02	.05658	.05728	.05585	.05590	.05548	.05562
5	154.56	.05718	.05785	.05483	.05646	.05458	.05618
6	158.00	.05998	.05914	.05810	.05772	.05750	.05743
7	158.72	.05983	.05941	.05768	.05798	.05755	.05769
8	161.69	.06102	.06052	.05950	.05906	.05900	.05877
9	167.71	.06408	.06277	.06085	.06126	.06052	.06096
10	211.58	.08048	.07919	.07760	.07729	.07743	.07691
11	221.41	.08350	.08287	.08018	.08088	.07990	.08048
12	221.50	.08268	.08291	.08090	.08091	.08065	.08051
13	222.44	.08230	.08326	.08045	.08126	.08025	.08086
14	222.69	.08278	.08335	.08097	.08135	.08090	.08095
15	227.50	.08488	.08516	.08298	.08310	.08242	.08270
16	238.69	.08730	.08934	.08728	.08719	.08655	.08676
17	239.22	.08985	.08954	.08680	.08739	.08652	.08696
18	257.81	.09532	.09650	.09450	.09418	.09400	.09371
19	274.23	.10373	.10264	.10018	.10018	.10073	.09968
20	276.95	.10445	.10366	.10068	.10117	.10035	.10067
21	281.19	.10468	.10525	.10265	.10272	.10218	.10221
22	287.62	.10810	.10766	.10505	.10507	.10470	.10455
23	298.46	.11105	.11171	.10838	.10903	.10785	.10849
24	302.28	.11408	.11314	.11013	.11042	.10965	.10988
25	305.64	.11428	.11440	.11175	.11164	.11100	.11108
26	320.94	.11870	.12013	.11695	.11724	.11648	.11661
27	333.87	.12457	.12497			.12175	.12136
28	335.07	.12530	.12542	.12148	.12240	.12088	.12180
29	335.07	.12515	.12542	.12262	.12240	.12327	.12180
30	335.21	.12550	.12547	.12198	.12245	.12160	.12185
31	341.30	.12895	.12775	.12515	.12468	.12500	.12406
32	353.41	.13253	.13228	.12910	.12910	.12805	.12846
33	354.63	.13220	.13236	.12972	.12955	.12915	.12891
34	391.07	.14425	.14638	.14298	.14286	.14200	.14215
35	396.46	.14880	.14839	.14550	.14483	.14470	.14411
36	397.11	.14913	.14864	.14458	.14506	.14385	.14435
37	397.60	.15038	.14882	.14545	.14524	.14530	.14453
38	398.25	.14990	.14906	.14513	.14548	.14438	.14476
39	416.23	.15563	.15579	.15223	.15205	.14902	.15130
40	419.57	.15690	.15705	.15342	.15327	.15228	.15251
41	436.47			.15950	.15944		
42	436.47	.16457	.16337	.16020	.15944	.16007	.15866
43	472.16	.17650	.17673	.17144	.17248	.17118	.17163
44	472.21	.17408	.17675	.17265	.17250	.17187	.17165
45	473.37	.17733	.17718	.17320	.17292	.17158	.17207
46	473.92	.17723	.17739	.17263	.17312	.17262	.17227
47	473.96	.17955	.17740	.17443	.17314	.17280	.17228

TABLE II. *Data on nitrogen at 0°C.*

No.	Pressure (cm Hg)	Δx_1(cm)4811A (obs.)	(calc.)	Δx_2(cm)5893A (obs.)	(calc.)	Δx_3(cm)6362A (obs.)	(calc.)
1	150.04	0.06123	0.06185	0.06053	0.06062	0.06023	0.06029
2	163.11	.06665	.06723	.06550	.06590	.06510	.06554
3	233.94	.09623	.09643	.09383	.09451	.09350	.09400
4	239.52	.09868	.09873	.09652	.09677	.09610	.09624
5	308.42	.12688	.12713	.12465	.12460	.12373	.12392
6	327.09	.13535	.13483	.13180	.13214	.13102	.13142
7	386.60	.15938	.15936	.15625	.15619	.15553	.15534
8	453.29	.18683	.18685	.18360	.18313	—	—
9	453.79	—	—	—	—	.18275	.18233

Computations and Results

The calculated values in the above tables were computed from the most probable values of $\Delta x/p$ obtained from an analysis of the data by least squares. The observed values of Δx were also plotted against pressure on a paper 50×90 cm of which Fig. 5 is a reduced photograph showing graphically

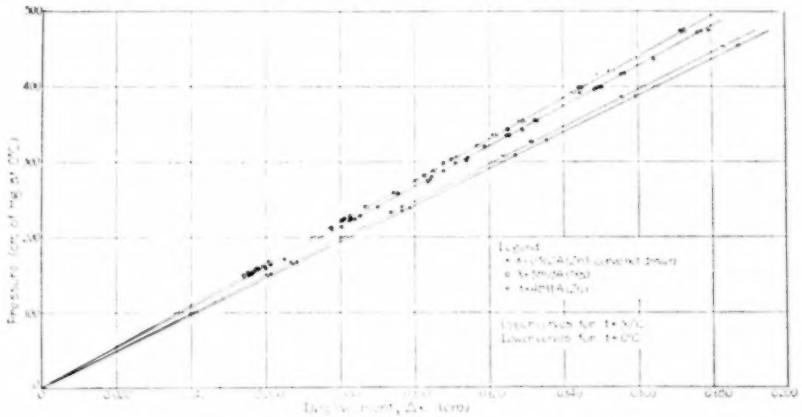

Fig. 5. Experimental results for nitrogen.

the linearity of the points. Values for B_0 were computed by Eq. (11) for the ranges 4811A–5893A and 4811A–6362A at each temperature.

	Dispersion ($B_0 \times 10^{16}$)	
Range (*A*)	*0°C*	*30°C*
4811–5893	1.90	2.08
4811–6362	1.87	1.95 Mean 1.95

Mean value of B (at 760 mm pressure) $= 1.48 \times 10^{-14}$. Although there is a very good agreement between the two values obtained at the zero temperature, the precision of the measurements is not sufficient to detect by experiment any temperature effect. For this reason the arithmetic mean of all the B values was used in the calculation of refractive index by Eq. (6). The values of $(\mu - 1)$ thus obtained at three different wave-legnths were reduced to values corresponding to infinite wave-length, by the Cauchy relation

$$(\mu - 1)_\infty = (\mu - 1) - \frac{B}{\lambda^2} \cdot \tag{14}$$

The values at 30° were reduced to zero degrees by Eq. (13) producing the results shown in the last row of Table III. The values in parentheses were obtained by reducing the 30° data to zero degrees for purposes of comparison with the data obtained directly at 0°.

Significance of Results and Conclusions

Observation of the graph Fig. 5, and a least square analysis of the data show that the displacements Δx follow very closely a linear relation with pres-

TABLE III. *Calculated results.*

	Temperature 0°C			Temperature 30°C		
	4811A	5893A	6362A	4811A	5893A	6362A
$\Delta x/p$	0.0004122	0.0004040	0.0004018	0.0003743	0.0003653	0.0003635
Prob. Error	±.0000001	±.0000001	±.0000001	±.00000005	±.00000003	±.00000003
$\Delta x/e$	.0003124	.0003061	.0003045	.0002836	.0002768	.0002755
$2B/\lambda^2$	.0000128	.0000085	.0000073	.0000128	.0000085	.0000073
$(\mu-1)$	.0002996	.0002976	.0002972	.0002708	.0002683	.0002682
	(.0002999)	(.0002973)	(.0002973)			
$(\mu-1)_\infty$	.0002932	.0002933	.0002935	.0002644	.0002640	.0002645
$(\mu-1)_{\infty,0}$	.0002932	.0002933	.0002935	.0002935	.0002930	.0002936
		Mean	.0002933			
			±.0000002			

sure. The origin lies just as well on the lines as any of the observed points. This linearity means, as was pointed out before, that the Lorentz-Lorenz relation holds, and that dispersion is a linear function of pressure for nitrogen up to seven atmospheres. Theory suggests an inverse linear dependence of dispersion on temperature although the data do not indicate it. The values of B_0 differ from the mean on the average by only 6% whereas an inverse linear temperature function would call for values at 0° to differ from values at 30° by 10%. In this connection it might be noted that Meggers and Peters data Fig. 2 show no consistent temperature effect.

On the other hand the precision of the measurements is not great enough to guarantee B_0 to better than 8%. The micrometer settings can be made to one third of one percent. At one atmosphere Δx is accurate to this degree, but as the pressure increases, so does the accuracy. Analysis shows the errors in the slopes to be less than one part in four thousand, yet due to the fact that B_0 depends on the difference between two slopes observed under different circumstances it is probably not accurate to better than approximately 8%. An error of 8%, however, in B_0 corresponds to an error of less than 0.3% in $(\mu-1)$. On the other hand a survey of the literature indicates that B_0 for a gas has not previously been recorded and that B has not been very accurately determined.

This information on dispersion enables one to reduce observations of refractive index at any wave-length to a value for infinite wave-length or zero frequency, and for this reason the results are probably of most value. It is, of course, the refractive index at zero frequency which should be compared by Maxwell's law with the dielectric constant for a gas. It will be noted that in Table III the values of $(\mu-1)_{\infty,0}$ all check to a remarkable degree, thus further emphasizing the degree of precision of the observations.

The above analysis of the data shows that the accidental errors in the results amount to less than one third of one percent. The only constant error of any consequence which affects them is the possible error in the pitch of the micrometer screw, which hitherto has been assumed to be 0.5 mm. To eliminate this uncertainty a very careful calibration of the screw was made over the range actually used by counting Michelson fringes as before described. The mercury 5461A line, filtered by screens provided by Cooper Hewitt, was

used for a source, and fringes were counted over a 2 mm range. The results of ten separate counts over a whole turn (approximately 1827 fringes per turn) indicated that the irregularities of the screw were negligible and that the actual value of the pitch was 0.499 mm in this range, making $\mu_{\infty 0}$ equal to 1.0002927 instead of 1.0002933, and limiting the absolute error to one third of one percent. As a result μ^2 is equal to 1.000585 to be compared with Zahn's[12] value of ϵ equal to 1.000581 which he estimates to be accurate to one percent in $(\epsilon-1)$. The difference between these values amounts to approximately two thirds of one percent in $(\epsilon-1)$.

Incidentally it might be observed that the values for the index of refraction at a pressure of one atmosphere and a temperature of zero degrees at the three wave-lengths are as follows, corrected for the screw error:

Refractive Index of Nitrogen at 0°C and 760 mm Pressure

λ	4811A	5893A	6362A
μ	1.0002991	1.0002969	1.0002967

These values are in accord with previously published results obtained by different methods.

ACKNOWLEDGMENTS

This problem was suggested by Professor F. G. Keyes, Director of the Research Laboratory of Physical Chemistry at the Massachusetts Institute of Technology, and at the time the work was started, also Consulting Professor of Physics at Brown University. To Professor Carl Barus and Professor C. W. Miller, the writer is also indebted for assistance in various matters, particularly with reference to the displacement interferometer. The writer is, however, especially grateful to Professor A. deF. Palmer for his sustained interest and help in supervising the whole problem.

[12] C. T. Zahn, (International Critical Tables, vol. 6, p. 78.); also Phys. Rev. **24**, 400 (1924).

ON THE ULTRAVIOLET ABSORPTION SPECTRUM OF ACETYLENE*

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Abstract

It is shown that the ultraviolet absorption bands reported previously do not belong to acetylene but to some impurity. In pure acetylene, bands are observed only below 2400A. These bands are arranged into three progressions and their interpretation is proposed using information from the work on the infrared absorption spectrum of acetylene.

THE more recent investigations of the infrared absorption spectrum of acetylene have revealed a very simple structure of these bands[1] leading to the conclusion that the acetylene molecule has all four atoms in a straight line and is therefore characterised by rotational quantum levels typical of a diatomic molecule. It seemed promising to undertake a study of the ultraviolet absorption spectrum of acetylene in the hope that here also a more complete analysis could be achieved than is possible with the majority of polyatomic molecules. Not much work has been done on acetylene spectrum. Stark and Lipp[2] investigated the absorption spectrum in gaseous acetylene and reported bands between 2200–1900A without giving their wave-lengths. On the other hand, Henri and Landau[3] studying gaseous acetylene and using a 40 cm absorption tube found numerous bands at longer wave-lengths which they divide into three groups: bands with fine structure from 3157 to 2872A; partly sharp, partly diffuse bands in the region 2960 to 2495A and bands with fine structure between 2327 and 2236A.

In the present work absorption tubes up to 2 meters length were used. As a light source a hydrogen discharge tube was employed and the photographs taken first with a medium size and later with a large Hilger El spectrograph. Acetylene was taken from a small "Prestolyte" tank and was purified[4] by passing it slowly through a series of five Milligan wash bottles containing water, chromic acid in sulfuric acid, mercuric chloride in hydrochloric acid, copper nitrate in nitric acid and sodium hydroxide solutions, respectively. It was finally dried over phosphorus pentoxide.

Acetylene thus purified was found to be completely transparent down to 2400A and even using plates of the most contrast no traces of the some 70 bands reported by Henri and Landau could be detected. Exposures were also made with acetylene prepared from commercial calcium carbide and only crudely purified. Here a continuous absorption was found extending from

* Contribution #18 from the Loomis Laboratory.

[1] Levin and Meyer, Journ. Opt. Soc. Am. **16,** 137 (1928); Hedfeld and Mecke, Zeits. f. Physik **64,** 151 (1930); Mecke, Ibid. **64,** 173 (1930).

[2] Stark and Lipp, Zeits. f. phys. Chem. **86,** 36 (1913).

[3] Landau, C. R. **156,** 697 (1913).

[4] Beilstein. Handbuch d. Organischen Chemie, Vol. 1.

shorter wave-lengths up to about 3200A. It became evident that the spectrum described by Henri and Landau is due to some impurity and a further search for it was abandoned.

In the short ultraviolet a number of bands was observed. The wave numbers of the band heads will be found in Table I. Most of these bands are extremely faint and, in addition, a continuous absorption sets in at about

TABLE I. *Absorption bands of acetylene.*

1st progression		2nd progression		3d progression	
Wave number	Estimated intensity	Wave number	Intensity	Wave number	Intensity
42056	1	42961	1	44084 44100	5
42156	1	43074 43088	5	44205 44220	3
42433	1	43184 43199	3	44304 44318	3
42504	1	43290	1	44475	3
42648 42660	2	43441 43455	2		
42698 42712	3	43502 43518	2		
		43674 43687	10		
		43730 43744	10		
	Unassigned	44441 (3) and 44514 (3)			

2300–2350A and becomes so strong below 2250A that not more than 5 percent of the light of this wave-length is transmitted by a 2 meter layer of acetylene at 760 mm pressure. In view of this, no great precision can be claimed for the wave-length determinations, the error for the fainter bands being probably of the order of 5 cm^{-1}. The bands are shaded to the red and have double heads with a spacing of about 14 cm^{-1}. The resolution obtained was not sufficient to show rotational structure near the heads, but in the stronger bands (43674–43687 and 43730–43744 cm^{-1}) evidence of structure was observed some distance away. Even in this spectral region the bands reported by Henri and Landau must be attributed to impurities for, with the exception of 42961 cm^{-1}, none of them coincides with the now observed ones within less than 20 cm^{-1}. It is of course not excluded that the bands here reported are also to be ascribed to impurities. On the other hand every precaution exercised in purifying acetylene seemed not to effect them in the least.

The bands recorded in Table I can be arranged into three progressions at a distance of 1032 and 1012 cm^{-1} apart. These progressions can be described by the same equation, as will be best seen from Fig. 1 in which the wave-numbers of the shorter wave-length band heads (after subtracting from the second and third progressions 1032 and 2044 cm^{-1}, respectively) have been plotted against an arbitrary number v. A quadratic expression in v is, however, certainly insufficient to describe the progressions since they show a pronounced convergence for the lowest values of v, but for $v=5$ to 10 the bands become nearly equidistant. It is also unusual that both the first and the second progression break off abruptly after the strongest band ($v=10$). A tenta-

tive interpretation of the observed regularities is possible using information obtained from the vibration-rotation bands of acetylene. Mecke[1] has pointed out that a normal acetylene molecule possesses three modes of vibration parallel to the molecular axis, of which one is optically active in the infrared ($\nu_2 = 3312.83$ cm^{-1}), and two transversal vibrations of lower frequencies ($\theta_1 = 729.27$ and $\theta_2 = 1328.5$). It is quite likely that the three progressions here reported are due to transitions into successive quantum states of either of the longitudinal vibrations of the excited molecule—probably ν_2—while the bands within each progression are caused by transitions to successive quantum states of one of the transversal modes of vibration. Against this interpretation speaks mainly the abnormal intensity distribution within each progression, but perhaps this, as also the rather unusual law of binding force necessary to account for the observed spacing of the bands can be attributed to the transversal character of the vibrations involved.

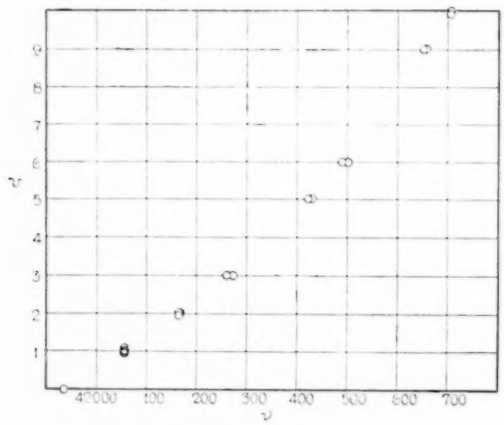

Fig. 1. A plot of three progressions of the acetylene bands.

Although a calculation of the fundamental frequencies of the excited molecule is impossible at present, it can be pointed out at least that they are probably lower than those of the normal molecule, the bands being shaded to the red, thus indicating a larger moment of inertia and presumably weaker binding force in the excited molecule. Assuming that the rotational structure of the acetylene bands is similar to that of diatomic molecules and that the double band heads are due to R and Q branches, one can estimate from their separation and the known moment of inertia of the normal molecule,[5] the moment of inertia of the excited one. One obtains thus $25.6 \pm 0.3 \times 10^{-40}$ gm cm^2 as compared with 23.509×10^{-40} gm cm^2 of the normal molecule.

The origin of the continuous absorption, mentioned in the beginning of this note, remains uncertain. It is likely, though, that this absorption is due to residual traces of impurities, because Stark and Lipp[2] were able to observe absorption bands down to 1900A.

It is a pleasant duty to thank Mr. Alfred L. Loomis for his kind permission to work in the Loomis Laboratory and for the facilities put there at my disposal.

[5] Assuming that the head of the Q branch coincides with the zero line of the band, that the equation of the R branch is $\Delta\nu = (B' + B'')K + (B' - B'')K^2$ and that for its head $K = -B' + B''/2(B' - B'')$.

SPACE CHARGE vs. IMAGE FORCE IN THERMIONIC EMISSION

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(Received December 7, 1930)

Abstract

This paper is a joint report on work by the two authors, preliminary to further and more detailed publication on different phases of the problems here discussed. Attention is called to certain deficiencies in the Schottky image force as an explanation of the thermionic work function, both on theoretical and experimental grounds. Evidence is advanced to show that space charge is more important than image force over most of the region in which the image force is thought to be valid. It is further shown that space charge with Fermi-Dirac statistics is able to account for observed phenomena of thermionic emission, including the effect of external fields. Numerical examples indicate some of the consequences of this point of view.

THE most commonly accepted explanation of the thermionic work function is that which attributes the major part of it to the "image" force of the electron (Debye, Schottky, Langmuir). Schottky[1] gave an analysis to show that space charge and structure effects, in the layer outside that in which the surface atomic structure of the emitting body plays an important part, were negligible compared with the image force. To do this it was necessary to estimate the electron density in this layer; this he calculated on the assumption that the image force law was correct, showing that under these conditions space charge and structure effects were negligible.

On the other hand, if the assumption is made that the potential and thereby the charge density in the above-mentioned layer is determined by space charge instead of by image force considerations then it may be shown that the space charge is more important than the image force. Thus we must look further for evidence by which to choose between the two explanations. As a numerical illustration of the difference involved, a calculation of the field intensities from these two contrasting standpoints at various distances from a plane tungsten cathode at 2300°K gives the following results in the absence of external field. E_i being the field intensity in v/cm computed from the image force, E_s that computed for the space charge field, we have, at a distance of 10^{-7} cm from the cathode, $E_i = 3.6\times10^6$, $E_s = 4.0\times10^6$; at a distance 5×10^{-7} cm, $E_i = 1.4\times10^5$, $E_s = 8.0\times10^5$; at 10^{-6} cm, $E_i = 3.6\times10^4$, $E_s = 4.0\times10^5$. Thus, for distances from the surface greater than about 10^{-7} cm, the field intensity computed from space charge considerations is greater than when computed by the image force, and this difference is accentuated as the distance from the surface increases. The concentration of electrons shows a similar behavior. The space charge treatment of course assumes a continuous distribution of electricity whereas the image force assumes the escape of only

[1] Schottky, Phys. Zeits. **15**, 872 (1914).

one electron. As Schottky correctly points out, since in practice we find a situation intermediate between these two view-points it is necessary to examine which of these two extremes is most nearly approached in practice.

A reconsideration of this matter seems desirable, not only in view of the above fact, but also because the earlier work on the subject postulated a Maxwell distribution law in the electron atmosphere. Since the advent of the Sommerfeld electron theory it becomes desirable to find whether the degeneracy of the electron gas at and presumably also near the emitting surface affects the analysis. Among the chief reasons for the general acceptance of the image force explanation have been (1) the observed constancy of the work function for a given pure metal under moderate applied accelerating fields, i.e. the thermionic saturation current, (2) for pure metals the fairly accurate verification of the variation of thermionic current with stronger applied fields (Schottky effect) as computed by Schottky, (3) the estimation of a reasonable value for the work function, with the opportunity of characteristic values for different materials. Its chief drawback at the present time appears to be in its application to emission from coated filaments, i.e. in failing to account for lack of saturation and for the Schottky effect observed in these cases without complicated, rather *ad hoc*, assumptions.

It is clear therefore that before any superiority may be pointed out a space charge analysis should also lead to confirmation with experiment in the above directions. In deriving the thermionic current, extending through the Schottky effect, by use of space charge considerations alone, if Poisson's equation is combined with a Maxwellian distribution of electron velocities the results show definite lack of agreement for pure metals. But if a Fermi distribution is substituted for the Maxwellian, the calculated dependence of current on temperature and applied field is found to be in reasonable agreement with experiment. This may be taken to indicate the constancy of the work function calculated by this method, but it seems more reasonable to deal with directly observable quantities, and compare calculated with observed currents under varying conditions. Furthermore, the variation of the current with the nature of the emitter can be explained more readily by space charge than by image force. According to the latter the field becomes constant within a certain critical distance from the surface, this distance being so chosen that the total work function thus calculated agrees with the experimental value. As justification for this step it is pointed out that close to the surface structure effects would be important. Now, assuming the space charge method valid right down to the surface, different currents from different metals are explained by the different electron concentrations within the metals, and the work function comes out to be of the correct order of magnitude according to the Sommerfeld theory (i.e. $= W_a$) without further assumption. This leads, incidentally, to a simple explanation of the observed interdependence of A and b, the constants in the usual thermionic emission equation. Moreover it appears that the use of Fermi-Dirac statistics takes account, in part at least, of structure effects. For a simple calculation will show that an electron gas is degenerate when the electrons are so closely crowded that the mutual po-

tential energy of neighboring pairs is large compared with their kinetic energy of thermal agitation, and obeys classical laws when the potential energy is small. For example, in an electron gas at 1600°K for which the Fermi A is 10^{-3}, i.e. the gas nearly in the classical state, the average kinetic energy is roughly twenty times the potential.

Thus it appears that the space charge calculation based on Fermi-Dirac statistics is a more proper method of dealing with pure metals, and it may be further pointed out that the same calculation based on classical statistics, assuming a low electron concentration in the emitter, seems capable of explaining results obtained with coated filaments. This latter point is being investigated further.

Following out their preliminary investigations on this subject[2] the authors have carried the work further, one of them taking exclusively the problem of deducing the thermionic current under applied accelerating fields from zero to fields operative in the Schottky effect, the other treating exclusively the case of the statistical equilibrium resulting under retarding applied fields (including zero). Both deal only with the case of an infinite plane emitting surface.

The former (R. S. Bartlett) has handled the problem after the manner of Langmuir and Fry by postulating the emission of electrons with velocities distributed according to the Fermi statistics, these electrons being subject only to the space charge field, and obeying the law of continuity of current. This calculation is rendered troublesome by the necessity of graphical computation of the Fermi integrals for the region of transition from a degenerate to a classical state, and thus are not at present in a form suitable for detailed report in this communication. The general results have been mentioned above, and appear encouraging, especially in pointing the way to a single explanation of thermionic currents from zero applied field up through fields of Schottky intensity.

The latter author (A. T. Waterman) starting with Poisson's equation and the Fermi analogue $A = A_0 e^{-W_i/kT}$ of the familiar Boltzmann relation, has evaluated the potential and the electron concentration at points outside the surface, where the atmosphere is in the classical state (assuming a degenerate state within the metal). In the presence of a retarding field E the solution takes different forms on either side of a critical distance, which, expressed in terms of potential difference between this distance and the surface, is $V' = 2 \log E/\beta$ where $\beta^2 = 16\pi(2\pi m)^{3/2}h^{-3}(kT)^{5/2}\, e^{W/kT}$ where W is Sommerfeld's W_i, the thermodynamic potential of the electrons within the body. Thus for $V > V'$ (thus including zero field), V (at distance x) $= -2kT/\epsilon \log (1 + \epsilon\beta/2kT\, x)$ and the number of electrons per cc, $n = kT/2\pi\epsilon^2 x^2$. For $V < V'$, $V = -kT/\epsilon(\epsilon Ex/kT - \log 4E^2/\beta^2)$ and $n = E^2/2\pi kT\, e^{-\epsilon Ex/kT}$. It will be noted that on the space charge conception the concentration of electrons in equilibrium with a hot body depends only upon the distance and the temperature,—in particular not upon the nature of the hot body, for distances at which the electron gas is in the classical state (roughly $> 10^{-7}$ cm). Examination as to

Abstracts 52, 53 Phys. Rev. **35,** 668 (1930).

the above-mentioned critical distance introduced by an applied retarding field, which is a measure of the distance inside which the applied field has no appreciable effect on the electron atmosphere, results in the observation that applied retarding fields of the order existing in cold extraction currents would control the electron atmosphere extending into the region where the electron gas becomes degenerate very near the surface. A prediction of the work is that the ratio of the anode potentials which would give equal thermionic currents under zero applied field from two different emitters under otherwise identical conditions (temperature and geometry of tube) should be $V_1/V_2 = \log \beta_1/\log \beta_2$. If Sommerfeld's electron theory is assumed, then this ratio should show appreciable difference from unity, e.g. $V_{Ca}/V_W = 0.65$; on the classical theory the ratio should always be very nearly unity, e.g. $V_{Ca}/V_W = 0.98$ in the instance quoted.

ORIGIN OF THE AXIAL ROTATION OF THE SUN

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(Received December 24, 1930)

Abstract

In earlier papers the solar atmosphere has been shown to rotate rapidly as a result of the electric and magnetic forces which act on the atmospheric ions. At the equator the ionized layers move much faster than the solar surface proper but this relative movement decreases with latitude in the manner required by observation. The fast moving atmospheric layers transfer momentum to the sun proper and its rotation is a necessary consequence of the observed atmospheric motions. The resultant torque is calculated and the angular acceleration shown to be adequate to account for the observed rotation.

THE almost universal prevalence of rapid rotation in celestial objects has been pointed out again and again by astronomers, but the origin of such rotation has been somewhat of a mystery. Celestial mechanics shows how a star may rotate faster and faster as it contracts due to condensation, but gives no clue as to the origin of the initial rotation, which must be comparable to the final value. In an important paper Jeans[1] discussed the relation of the radiation of matter to the rotation of a star and he concluded that the radiation of the star carries off angular momentum in such a way that it "generally lessens the angular momentum per unit mass of the star." He thus pointed out that the common assumption of a constant angular momentum for a radiating star is invalid. We must, therefore, for the most part, attribute the observed high rotational velocities to changes in the mean density together with an initial rotation of the star or else to some phenomena not adequately described by celestial mechanics.

In a series of papers which deal with the electromagnetic effects operating in the solar atmosphere, it has been shown[2] that the anomalies of the sun's apparent rotation arise from an electromagnetic drift of the ionized atmosphere and are not directly dependent on the motion of the surface of the sun proper. It was shown that the solar atmosphere rotates in the same direction as the sun proper but *much faster* and that this superposed drift results from the interaction of the atmospheric ions with the observed solar magnetic field and a radial electric field.

The sun may be thought of as a highly viscous mass of gas bound together by gravitational, radiative and electromagnetic forces in such a manner as to form a semi-rigid system. Radiative viscosity acts in such a manner as to equalize differences of rotation in different layers but Jeans[1] calculations indicate that these forces alone are incapable of equalizing the rotations in a time

[1] Jeans, Monthly Notices R. A. S. **86,** 328, 444 (1926).
[2] Gunn, Phys. Rev. **35,** 635 (1930); **36,** 1251 (1930).

so short as the age of the sun. The sun's observed magnetic field is asymmetric with respect to the axis of rotation and hence shells of highly conducting solar matter rotating about the geographical axis will induce electromotive forces and systems of electrical eddy currents which will strongly oppose motion relative to the magnetic field. A quantitative calculation of the electromagnetic damping forces due to axial slippage is difficult but we have seen that in regions of high electrical conductivity the forces due to radial contraction are tremendous[3] and it seems probable that electrical forces contribute in an important manner to the apparent viscosity of the sun. To be sure that these forces have time to act we must examine whether the electrical time constant of the induced current circuit is sufficiently short compared to the solar age for current systems to build up. It is convenient to note that the largest time constant τ possible is the constant for the whole sun which is given by $\tau = L\Sigma$ where L is the effective inductance and Σ is the total conductivity. On making this calculation it results that τ is of the order of 10^9 years. This is very small compared to the age of the sun and we assume that inductive effects are unimportant.

On account of the high effective viscosity of the interior it seems satisfactory to assume as a first approximation that the sun is a semi-rigid body rotating on its axis at a period corresponding to the observed period of rotation of its magnetic pole. With the solar radius known it is evident that the peripheral velocity of any point on the solar surface proper can be calculated. The difference between this velocity and the velocity of the same point as determined by an earth-bound observer watching the solar atmosphere is evidently the atmospheric drift velocity. This superposed drift velocity $\boldsymbol{U}$ of the ionized atmosphere is readily determined from observation or can be calculated from[2]

$$\boldsymbol{u} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2\left(1 + \left(\frac{R}{\lambda}\right)^2\right)} \tag{1}$$

where E and B are the electric and magnetic field intensities, λ the mean free path of the ions, and R the radius of the helix generated by an ion as it spirals around the impressed magnetic field. This radius is calculated from

$$R = \frac{mV}{Be} = \frac{(2mkT)^{1/2}}{Be} \tag{2}$$

where m is the mass of the ion, V the component of velocity perpendicular to B and e the ionic charge.

It is clear from Eq. (1) that the drift motion of the atmosphere drops abruptly as the ion pressure increases to such a value that the free path becomes of the order of R. Thus a more or less sharply defined transition layer exists between the atmosphere and the solar surface proper. In this critical transition region between the sun and its atmosphere there is a constant

[3] Gunn, Phys. Rev. **35**, 107 (1930).

interchange of momentum, and the faster moving layers transfer their momentum to the slower. At the surface of the sun the temperature is low and the radiative and electromagnetic viscosities are small compared to the molecular viscosity, so that we need only consider the forces due to the latter. The tangential force applied to the periphery of the sun by the more rapidly rotating atmosphere is

$$F = \eta u \frac{A}{S} \tag{3}$$

where η is the mean coefficient of viscosity of the transition layer, u the mean difference in velocity of the layers, A the effective area in contact and S their separation. To a sufficient approximation we may assume that in the transition layer the coefficient of viscosity is independent of the magnetic field and write

$$\eta = \frac{(3ZmkT)^{1/2}}{2(2)^{1/2}\pi\sigma^2} \tag{4}$$

where m is the mass of the hydrogen atom, Z the mean atomic weight of the ions composing the layer, k the Boltzmann constant, T the absolute temperature, and σ the kinetic theory diameter of the ions. To a sufficient approximation we may assume that the effective contact area A is a ring around the equator of width R where R is the radius of the sun and that the mean velocity difference is not appreciably different from the equatorial value. The torque Q applied to the solar mass is

$$Q = I\alpha = \frac{(3ZmkT)^{1/2}}{(2)^{1/2}\sigma^2} \frac{u}{S} R^3 \tag{5}$$

where I is the effective moment of inertia and α is the angular acceleration. Assuming that the solar density is uniform we get from Eq. (5)

$$\alpha = \frac{5(3ZmkT)^{1/2}}{2(2)^{1/2}\sigma^2} \frac{uR}{SM} \tag{6}$$

where M is the mass of the sun. Taking from earlier work $Z=3.3$, $m=1.66\times10^{-24}$ gm, $k=1.37\times10^{-16}$, $T=6{,}000$, $\sigma=10^{-8}$ cm, $R=7\times10^{10}$ cm, $M=2\times10^{33}$ gm, $u=5\times10^{4}$ cm/sec and $S=5\times10^{6}$ cm we find $\alpha=2.3\times10^{-26}$ rad/sec.[2]

We have seen in an earlier paper[4] that the solar magnetic field is maintained by a system of electrical currents which flow in rings about the sun's axis and that this system of currents is not primarily related to the rotation, but to the sun's radial and axial symmetry. This circumstance suggests that the magnetic field and hence motions of the solar atmosphere have not changed in order of magnitude for a considerable portion of the sun's life, which various estimates put at 7×10^{12} years or 2×10^{21} seconds. By multiply-

[4] Gunn, Phys. Rev. **34**, 335, 1621 (1929).

ing the angular acceleration α by the life time we find to a crude approximation that the final angular velocity of the sun should be 4.6×10^{-6} radians per second, if its initial rotation was negligibly small. The observed angular velocity, as computed from the motion of the magnetic pole is 2.3×10^{-6} radians per second. The agreement is rather better than one might expect from a calculation based on steady states.

Were allowance made for the increased radiation during the youthful period and a smaller moment of inertia used, due to the concentration of mass toward the center, then the elapsed time necessary to account for the observed rotation would be considerably reduced. This suggests that in the earlier stages of development the sun was unmagnetized and was a typical pulsating star.[3] It is clear that the solar rotation can be accounted for by the mechanical drag of the solar atmosphere on the sun proper if the velocity of the atmosphere exceeds that at the surface by 0.5 km/sec. This is the precise value required to explain the anomalous rotation of the sun; this having been adequately accounted for in an earlier paper by the author. The mechanism of rotation which we have considered doubtless has universal application in all stars having a magnetic field. It seems probable that this mechanism, which constantly supplies angular momentum to stellar systems, may assist in explaining certain puzzling facts in connection with spiral nebulae and binary systems. Numerical application to other star systems seems meaningless at the present time due to lack of data, and we can only point out that most other stars will behave like the sun. Ordinary mechanics has been found inadequate to explain the rapid rotation of heavenly bodies. The present investigation shows that the rotations can be accounted for by the momentum transferred to the body proper by its own highly ionized atmosphere. We have shown previously that the motion of the solar atmosphere is controlled by the magnetic and electric fields existing in the atmospheric layers and that these fields are probably controlled by the rate of radiation of matter. We may then conclude that the rotation of all heavenly bodies is intimately related to the dissolution of matter and in a static, non-radiating universe rotating systems would not exist.

THE NATURE OF THE CONDUCTIVITY OF INSULATING OILS*

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(Received December 23, 1930)

Abstract

It has been shown in a previous paper how the methods developed in the study of conduction of electricity through gases could be applied to determine the nature of the ions present in liquids and solids. In the present paper these methods have been applied to the measurements of the space charge set up in insulating oils by a steady electric field, which were recently performed by Whitehead and Marvin. The procedure is as follows: the shape of the curve shows that the current must be considerably below three-fourths of the saturation current. It is possible to fit the curve closely by assuming the Langevin constant to be $\beta = 36 \pm 10$. From this the number of ions present in the normal state and their mobility can be calculated, which turns out to be $3.5 \cdot 10^{-6} \pm 50\%$ cm/sec per volt/cm. With the assumption that the mobility is inversely proportional to the viscosity, the mobility of these ions in water would be $1.2\ 10^{-4}$ cm/sec per volt/cm or in the usual units 12. If one assumes that Lorenz's results for organic ions can be applied these ions should have approximately 110 atoms. From the boiling point of the oil one would conclude that the molecule of the oil contains about 45 atoms.

A NUMBER of papers[1] have recently appeared from the Engineering School of The Johns Hopkins University, investigating the conductivity of transformer oils. The last of these make possible an application[2] of Mie's[3] formulas to discover the properties of the ions.

We start with the space-charge distribution[4] in oil "B", when a potential of 1500 volts is applied between large condenser plates 2 cm apart. The distribution of potential is nearly symmetric around the center, proving that the mobility of the ions of opposite sign must be nearly equal. We are going to neglect whatever dissymmetry might be present.

Fig. 1 shows that the line is very nearly straight in the neighborhood of the center, i.e. the field very closely constant for quite a distance. It follows from this that the current must be considerably below $\frac{3}{4}$ of the saturation current and the approximate formulas have to be used which apply, according to Mie, to weak currents.

* Presented at The Third Annual Meeting and Conference, Committee on Electrical Insulation, Division of Engineering and Industrial Research, National Research Council, Bureau of Standards, November 7, 8, 1930.

[1] J. B. Whitehead and R. H. Marvin, A. I. E. E. Trans. **48,** 299 (1929); J. B. Whitehead, Jl. Franklin Inst. (IV) **208,** 453 (1929); J. B. Whitehead and R. H. Marvin, A. I. E. E. Trans. **49,** 641 (1930).

[2] K. F. Herzfeld, Phys. Rev. **34,** 791 (1929).

[3] G. Mie, Ann. d. Physik **13,** 857 (1904).

[4] An enlarged drawing of this figure, which has been used for Fig. 1 of the present paper, has been kindly put at my disposal by Dr. J. B. Whitehead.

The next step is the calculation of β(β^{-1} is Langevin's constant). For this purpose, one has to compare the field in the center and the field near the electrodes. The field in the center is

$$h_0 = 450 \pm 10 \text{ volt/cm}.$$

The field near the electrodes is not measured directly; from the drawing it was estimated to be $h_m = 2400$ volt/cm. This would lead to

$$c = \beta^{\beta/\beta-1} = \frac{h_m^2}{h_0^2} = \left(\frac{2400}{450}\right)^2 = 28.5.$$

Subsequently, $c = 40$ proved to be the value which suited the curve best, but calculations were also made with $c = 30$ for comparison, which turn out to fit the curve appreciably less.

$$c = 40 \text{ gives } \beta = 36.3, \ (2\beta)^{1/2} = 8.5.$$

It turns out that it is more useful for the actual calculation to put Mie's formulas into a form different from the one that had been given previously.

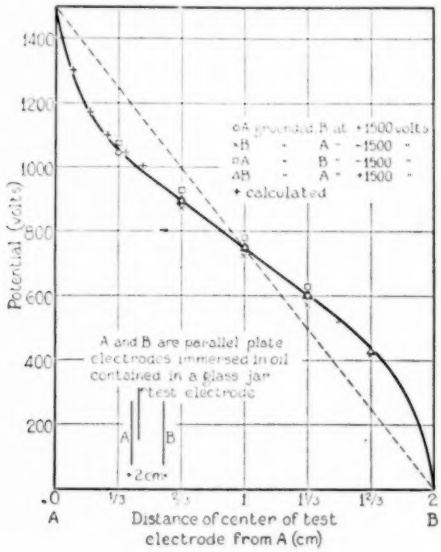

Fig. 1.

Calling, as was done there, x_K the location of the bend in the graph of the field, we measure the location in the condenser relative to x_K and find for the flat (central) part[5]

$$h^2 = H^2[1 + (c-1)(e^{-y_1} + e^{-y_2})] \tag{1}$$

$$y_1 = \frac{(c-1)(c-1-(2\beta)^{1/2})}{2\beta - c + 1}\left(\frac{c-1-(2\beta)^{1/2}}{c-1} - 2\frac{l-x}{l-x_K}\right) \tag{2}$$

in y_2 only $l-x$ has to be replaced by $l+x$.

[5] In formula (50) of the paper cited in footnote (2) $4l^2$ should be stricken from the denominator.

With the high value of β given above the exponent is always so large that the deviation from a constant $h=H$ for the field or from a straight line $V=-Hx+V_0/2$ for the potential is negligible in the range of this formula, that is within $-x_K<x<x_K$. That means $h_0=H$, there is no change in the amount of ions in the central part. In the central part Ohm's law is preserved with the initial conductivity. In the range between the electrodes and x_K, (1) holds, but there y_1 and y_2 have a different form from the one in the central part. y_2 as before is derived from y_1 by substituting $l+x$ for $l-x$. We have

$$y_1 = +(2\beta)^{1/2}\frac{c-1-(2\beta)^{1/2}}{2\beta-c+1}\,\frac{l-x}{l-x_K}\left(1+\frac{1}{2}\,\frac{c-1-(2\beta)^{1/2}}{2\beta}\,\frac{l-x}{l-x_K}\right). \quad (3)$$

But here too, with the large value of β, the coefficients are so large that $(c-1)e^{-y_1}$ is appreciable only for small values of $l-x/l-x_K$, and than e^{-y_2} with the argument $l+x/l-x_K$ is negligible. With this in mind, it is useful to measure, as Mie does, the distance from the electrode instead of the center and write

$$h^2 = h_0^2[1+(c-1)e^{-y_1}] \quad (1')$$

$$y_1 = (2\beta)^{1/2}\frac{c-1-(2\beta)^{1/2}}{2\beta-c+1}\,\frac{x}{x_K}\left(1+\frac{1}{2}\,\frac{c-1-(2\beta)^{1/2}}{2\beta}\,\frac{x}{x_K}\right). \quad (3')$$

The subsequent procedure is as follows:

One draws a curve of h as function of x/x_K, from $x/x_K=0$ (electrode) to $x/x_K=1$ (bend) and proceeds from thereon with $h=h_0$. The problem is to determine x_K, or to put it differently, to determine l/x_K.

We accordingly integrate graphically beyond the point $x/x_K=1$, taking into consideration the scale of our drawing and changing it continuously as we go on [because $V=\int h dx = x_K \int h d(x/x_K)$], until we get to a point where $V=V_0/2=750$ volts. This must be the center. To show a few examples, we get with different assumptions for the position of the bend:

$\frac{x_K}{l}$,	$\frac{8}{12}$	$\frac{10.5}{12}$	$\frac{11}{12}$
V	680	750	765.

It seems therefore safe to say that

$$\frac{x_K}{l}=\frac{7}{8}=0.88\pm 0.08.$$

The points calculated with this value and $c=40$ are marked with crosses in the figure. Formula (53)[6] gives then

$$\frac{j}{j_{sat}}=\frac{x_K}{l}\,\frac{(2\beta)^{1/2}(2\beta-c+1)}{(c-1)(c-1-(2\beta)^{1/2})}=0.2\pm 0.05.$$

[6] See footnote (2).

We put then the current $j=\delta 1.6\times 10^{-11}$ amp/cm² with δ probably near unity. That makes $j_{sat}=\delta 8\times 10^{-11}\pm 25\%$ amp/cm² or, if we consider that j_{sat} is the total amount of charge generated per cm², that is in a cylinder of $2l=2$ cm length, that gives the number of pairs of univalent ions, generated in 1 cc per second

$$q=\frac{j_{sat}}{2.2e}=1.25\times 10^{8}\cdot\delta\pm 25\%.$$

Furthermore, we have

$$\beta=\frac{8\pi K_e}{\alpha\epsilon}=40\pm 10.$$

If we want K in cm/sec/ volt/cm instead of per est. unit/cm, we have to divide by 300. It is this number that shall be meant by K from now on.

We assume $\epsilon=2.5\pm 10\%$

$$\frac{K}{\alpha}=2.8\times 10^{7}\pm 35\%.$$

Now $q/\alpha=n_0^2$ (n_0 being the equilibrium number of the ions of each sign which is present in the central part of the condenser even under the conditions of our experiment). Therefore

$$\frac{qK}{\alpha}=Kn_0^2=3.5\times 10^{15}\delta\pm 50\%.$$

On the other hand, the conductivity in the central part, when Ohm's law is valid gives

$$\frac{j}{450.2.e}=n_0K=1.1\times 10^{5}\delta$$

or

$$n_0=3.2\times 10^{10}\pm 50\%$$

and

$$K=3.5\times 10^{-6}\delta\pm 50\%\text{ in cm/sec/volt/cm}.$$

[7] The current density was measured on March 27 and found to be 1.6×10^{-11} amp/cm²; the space charge curve was taken April 4–6, a remeasurement of current on May 2 gave 2.9×10^{-11} amp/cm², the curves in Fig. 12 (in the last paper mentioned in footnote 1) give for the initial conductivity $5.1\times 10^{-14}\,h_0$ which, for 450 volt/cm, would amount to a current of 2.25×10^{-11} amp/cm². But during April a considerable manipulation with the oil makes the first given current more trustworthy. There is another fact pointing in this direction. The ratio of the initial field to the central field in the steady state should be equal to the ratio of the initial current to the steady current, if the conductivity (number of ions) remains unchanged in the central part. But in the later measurements the ratio of the currents is only 1.45 (private communication), while the ratio of the fields is 750/450 = 1.67. This is clear if in the meantime the number of ions has increased because then one would be further from saturation and the ratio of the two fields at the time of the second current measurements would be less.

On the other hand, the measurement on March 27 (Fig. 11) points to a ratio of currents 1.7.

To get a reasonable comparison for this value, one remembers that according to Walden,[8] the mobility of large ions in different solvents is inversely proportional to their internal friction η. Professor J. C. Hubbard, of this Laboratory, was kind enough to determine η for this oil and found it 0.35 (water 0.01), which would make the mobility in water 1.2×10^{-4} cm/sec/volt/cm or, in the usual units (multiplication with 96540 coulomb) equal to 12. $\pm$ 50%.

According to Lorenz[9] one could now estimate that an anion of mobility 12 contains about 110 atoms, an ion of mobility 18, 65 atoms.

M. F. Hamburger, of the Department of Electrical Engineering, has kindly measured the boiling point of the oil and found it to be about 290°C. Accordingly, one would estimate that the oil has molecules containing about 15 C-atoms, therefore about 45 atoms. If the ions are products of oxidation (last paper footnote 1) the anion should contain about two molecules (be the anion of an acid coming from the oxidation of a polymer of double molecular weight). The equal mobility of the action would have to be explained as due to the fact that the H^+ ion has two oil molecules (or a molecule of the above mentioned isomer) attached to it.

If the interpretation of Dr. Whitehead, that the increase in conductivity above the Ohm-value upon reversal is due to an ion of a mobility $K = 10^{-4}$ cm/sec/volt/cm is correct, this ion would turn out, with our present data, to be very quick. Multiplying namely with 10^5 (units!) and 35 (ratio of η of oil and water) we get 350 which is a speed only attained by H^+ in water. We might have to divide this by 2 or 3 as small anorganic ions move in water slower as expected, and reduce the estimated K to $1/2\ 10^{-4}$. Then we could still have the mobility of a small anorganic ion.

If this is accepted, then the picture is as follows: There are present a large number of slow, organic ions which are continuously generated and move on to the electrodes where they are discharged without appreciable accumulation They are responsible for the space charge and the long time phenomena.

Besides, there are quick, small anorganic ions which are not discharged at the electrodes, but accumulate there, without practically influencing the space charge, and are the cause of short time processes. Offhand, three origins could be imagined: (a) They are present as impurities, but are not supplemented continuously. That is impossible, because then upon reversal of the field the current would not go beyond the initial current of the original field. (b) They are generated continuously; then their number must be much smaller than the number of the slow ions, because the latter determine the space charge (β). (c) Their origin could be a secondary process on the electrodes (OH'?).

[8] P. Walden, Zeits. f. anorg. Chem. **113,** 85 (1920).

[9] R. Lorenz, Raumerfüllung u. Ionenbeweglichkeit, Leipzig, (1922) p. 79.

POTENTIAL DUE TO A BURIED SPHERE

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Abstract

A series solution of the problem dealing with the electrical potential due to a buried conducting sphere is given. The general method of attack consists in finding a solution of Laplace's equation $\nabla^2\Phi=0$, subject to proper boundary conditions. The condition requiring the vanishing of the normal derivative of potential on the surface of the half space is satisfied by placing an image sphere in an upper half space. The introduction of this image sphere also introduces the complication of the interaction of the potential due to two spheres. This is handled by means of spherical harmonics and the solution of the problem is made possible by means of a special transformation in spherical harmonics.

Introduction

MANY problems of this type have arisen recently in connection with the subject of geophysics. Since the introduction of electrical methods for ore exploration, there have been numerous attempts to find some means by

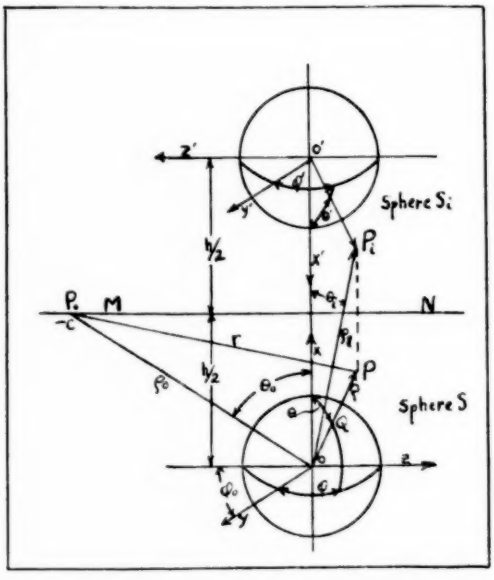

Fig. 1. Buried and image sphere.

which to predict the nature and scope of irregularities in homogeneity of structure beneath the surface of the earth, when point electrodes on the surface are maintained at definite potentials.[1] In most of these attempts, cer-

[1] A. Petrowsky, Phil. Mag. [5] **31**, 927 (1928); **28**, 334 (1928).

tain approximations have been made. In fact, in the more complicated problems, approximations must be made. The present problem of the sphere is worked out, making no approximations, as a first step toward finding out the allowable approximations which may be made in the more complicated problems, e.g. a buried ellipsoid or cube.

Assume a sphere of radius a and conductivity σ_1 buried in an infinite half space of conductivity σ_2 (see Fig. 1). At the center of the sphere, locate the common origin of a set of cartesian coordinates x, y, z and a set of polar coordinates ρ,θ,ϕ. Place an electrode at the common point $P_0(h/2, 0\text{-}c)$ and P_0 $(\rho_0, \theta_0, \phi_0)$. Note that $\phi_0=\pi/2$. The potential due to the electrode alone is $I/4\pi r\sigma_2$ in which r is the distance from P_0 to any point P and I is the current. It is desired to find the potential at every point in space due to the buried sphere S. Designate by u and U the inside and outside potential, respectively, due to the sphere alone. Then the functions u and U must satisfy Laplace's equation

$$\nabla^2\Phi = 0, \tag{1}$$

at all points in space, and are further subject to the conditions:

$$k\frac{\partial u}{\partial \rho} - \frac{\partial U}{\partial \rho} = -(k-1)\frac{\partial U_e}{\partial \rho}, \quad \rho = a, \tag{2}$$

$$u = U, \qquad \rho = a, \tag{3}$$

$$U = 0, \qquad \rho = \infty \tag{4}$$

and

$$\frac{\partial \Phi}{\partial x} = 0, \quad x = h/2, \tag{5}$$

in which U_e denotes the potential due to the electrode and k represents the ratio of conductivities σ_1/σ_2. Condition (2) results from the well-known equation:

$$k\frac{\partial \Phi}{\partial n_1} + \frac{\partial \Phi}{\partial n_2} = 0$$

in which Φ is the total potential and n_1 and n_2 are the normals pointing into the regions whose conductivity is σ_1 and σ_2 respectively. In the above equation

$$u + U_e = \Phi,$$

inside the sphere, while

$$U + U_e = \Phi$$

outside the sphere. Then since

$$\frac{\partial}{\partial n_1} = -\frac{\partial}{\partial \rho},$$

while

$$\frac{\partial}{\partial n_2} = \frac{\partial}{\partial \rho},$$

the last equation becomes

$$-k\left(\frac{\partial u}{\partial \rho} + \frac{\partial U_e}{\partial \rho}\right) + \left(\frac{\partial U}{\partial \rho} + \frac{\partial U_e}{\partial \rho}\right) = 0,$$

from which Eq. (2) follows at once. The complication which arises in the solution of the above problem is due to the boundary condition (5). In order to satisfy this condition, an image sphere S_i is placed in the upper half space, which is imagined filled with material of conductivity σ_2. Then, from symmetry, the normal derivative of potential will vanish on the boundary

$$x = \frac{h}{2}.$$

It may be seen that the problem can be viewed as either one in which a single sphere S is situated in an infinite half space, subject to the boundary condition (5), or a problem in which two spheres S and S_i, (S_i being the image in plane MN of S), are situated in an (whole) space. In the problem which includes the image sphere, the boundary condition (5) is automatically satisfied while at the same time, the complication arises that the interaction between the two spheres must be taken into account. Analytically, this results from the fact that, when the image sphere is included, the applied potential consists not only of U_e, the potential due to the electrode, but also U_i, the potential due to the image sphere. This new potential will be called U_a. According to the uniqueness theorem, the solution of the problem with the image sphere will also be a solution of the problem in hand, since such a solution satisfies Laplace's equation and boundary conditions (2), (3), (4) and (5). The solution of the problem, using the image sphere is the one here given.

The following solutions of Laplace's equation may be written immediately

$$u_1 = \sum_{m=0}^{\infty} \left(\frac{\rho}{a}\right)^m Y_m(\mu, \phi), \tag{6}$$

$$u_2 = \sum_{m=0}^{\infty} \left(\frac{a}{\rho}\right)^{m+1} Y_m'(\mu, \phi), \tag{7}$$

in which $Y_m(\mu, \phi)$ stands for the ordinary spherical harmonic and μ for $\cos\theta$. Because of condition (4) and because the potential of the sphere must be finite at $\rho = 0$, u_1 and u_2 are chosen as u and U, the inside and outside potentials respectively. If the coefficients A_{nm} and B_{nm}, in the above expressions, can be so chosen that u and U satisfy all the boundary conditions, the solution of the problem will have been found.

Eqs. (6) and (7) satisfy Laplace's equation, condition (4), and will satisfy condition (3) provided the coefficients in (6) and (7) be set equal, making

$$Y_m(\mu, \phi) \equiv Y_m'(\mu, \phi).$$

The next step is to apply condition (2). Before this is possible, a suitable expression for U_a must be found. U_a consists of two parts

$$U_a = U_e + U_i, \tag{8}$$

a part U_e due to the electrode and a part U_i due to the image sphere S_i. From symmetry, the potential at any point P due to sphere S_i is the same as the potential at point P_i due to sphere S, where[2] P_i is the image of P in the plane MN, i., e.,

$$U_i(P) \equiv U(P_i) = \sum_{m=0}^{\infty} \left(\frac{a}{\rho_i}\right)^{m+1} Y_m(\mu_i \phi_i). \tag{9}$$

Thus the expression for the potential U_a, at P, can be written,

$$U_a = \frac{I}{4\pi\rho_0\sigma_2} \sum_{m=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^m L_m(\mu_0, \phi_0, \mu, \phi) + \sum_{m=0}^{\infty} \left(\frac{a}{\rho_i}\right)^{m+1} Y_m(\mu_i, \phi_i), \tag{10}$$

in which L_m is a Laplacian, a special form of the spherical harmonic Y_m. Since $\phi = \phi_i$, ϕ_i may be replaced by ϕ. Hereafter D will be written for the constant term $I/4\pi\sigma_2$. The first member of Eq. (10) is the part of the potential due to the electrode, and the second member is the part due to the image sphere S_i. The coefficients in the second member of (10) are identical with the coefficients in the expressions (6) and (7) for the potential of the sphere S. The boundary condition (2) will be employed for determining the values of these coefficients which appear in (6), (7) and (10). This will be accomplished by substituting for u, U and U_a their values from (6), (7) and (10) in Eq. (2) and equating coefficients of like tesseral harmonics. Before this is possible, however, U_a must be expressed as a sum of tesseral harmonics with arguments μ and ϕ whose coefficients contain (besides the constants A_{nm} and B_{nm} to be determined) successive powers of ρ. The first term of (10) is already essentially in the desired form. The second term of (10) is at present, however, expressed in terms of ρ_i, μ_i and ϕ. It will be expressed, as desired, in terms of ρ, μ and ϕ by the establishment of a relationship of the type:

$$\sum_{m=0}^{\infty} \left(\frac{a}{\rho_i}\right)^{m+1} Y_m(\mu_i, \phi) = \sum_{m=0}^{\infty} f_m(\rho') Y_m(\mu', \phi), \tag{11}$$

and subsequently replacing ρ' and μ' by their values in terms of ρ and μ. When the second term of (10) has been thus transformed, the solution of the problem is then found at once by comparing on the two sides of Eq. (2) coefficients of like tesseral harmonics.

Now a relationship between tesseral harmonics, which makes possible a transformation of type (11), has been found. This relationship is

$$\frac{P_m{}^n(\mu_i)}{\rho_i{}^{m+1}} = \frac{(-1)^{m-n}}{(m-n)!}(1-\mu'^2)^{n/2} \frac{\partial^{m-n}}{\partial h^{m-n}} \frac{1}{h^n} \frac{\partial^n}{\partial \mu'^n} \sum_{k=n}^{\infty} \frac{\rho'^k}{h^{k+n+1}} P_k(\mu'), \tag{12}$$

[2] The subscript i is used throughout to refer to image points.

in which the quantities θ' and θ_i are related a shown in Fig. 1. Differentiating with respect to μ' and h, the result is

$$\frac{P_m{}^n(\mu_i)}{\rho_i{}^{m+1}} = \sum_{k=n}^{\infty} \frac{(m+k)!}{(m-n)!(n+k)!} \frac{\rho'^k}{h^{m+k+1}} P_k{}^n(\mu') = \sum_k \alpha_{knm}. \tag{13}$$

For the special case $n = 0$, (13) reduces to the form,[3]

$$\frac{P_m(\mu_i)}{\rho_i{}^{m+1}} = \sum_{k=0}^{\infty} \frac{(m+k)!}{m!k!} \frac{\rho'^k}{h^{m+k+1}} P_k(\mu') = \sum_k \alpha_{k0m}. \tag{14}$$

Substituting (13) and (14) in the second member of (10) one obtains

$$U_a = \frac{D}{\rho_0} \sum_{m=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^m L_m(\mu_0\phi_0\mu\phi) + \sum_{m=0}^{\infty} \left[A_{0m} \sum_k \alpha_{k0m} + \sum_{n=m}^{\infty} (A_{nm} \cos n\phi + B_{nm} \sin n\phi) \sum_k \alpha_{knm} \right].$$

By an interchange of indices m and k, one finds

$$U_a = \frac{D}{\rho_0} \sum_{m=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^m L_m(\mu_0, \phi_0, \mu, \phi) + \sum_{m=0}^{\infty} \left[\sum_k A_{0k}\alpha_{m0k} + \sum_{n=m}^{\infty} \sum_k (A_{nk} \cos n\phi + B_{nk} \sin n\phi)\alpha_{mnk} \right], \tag{15}$$

which is the desired result. It is now possible to apply the boundary Eq. (2). The values of u, U and U_a to be substituted in Eq. (2) have been given above. This boundary Eq. (2) involves the first derivative of u, U and U_a with respect to ρ. The second member of (15) for U_a, gives the potential in terms of ρ' and μ' at point P_i, due to the sphere S. From symmetry it may be seen, however, that if ρ' and μ' be replaced by ρ and μ, the second member of (15) will then give the potential at point P, due to the sphere S_i. If now the values of u, U and U_a be substituted in Eq. (2), ρ set equal to a, and coefficients of like tesseral harmonics equated, the following sets of linear equations result:

$$\alpha_m A_{nm} = K_{nm}\left(\frac{a}{\rho_0}\right)^m - \sum_{k=n}^{\infty} a_{nkm} A_{nk}, \tag{16}$$

$$\alpha_m B_{nm} = K_{nm}\left(\frac{a}{\rho_0}\right)^m - \sum_{k=n}^{\infty} a_{nkm} B_{nk}, \tag{17}$$

$$(n = 1, 2 \cdots \infty), (m = n, n+1, \cdots \infty),$$
$$(n = 0), (m, k = 1, 2 \cdots \infty),$$

in which

$$\alpha_m = \left[\frac{(k+1)}{(k-1)} + \frac{1}{m(k-1)}\right], \tag{18}$$

[3] See Whittaker and Watson, *Modern Anal.* 4th. Ed., p. 400, Ex. 7.

$$K_{nm} = \frac{2D}{\rho_0}\left(\frac{a}{\rho_0}\right)^m \frac{(m-n)!}{(m+n)!}\cos n\phi_0 P_m{}^n(\mu_0), \tag{19}$$

and

$$a_{nkm} = \frac{(m+k)!}{(m+n)!(k-n)!}\left(\frac{a}{h}\right)^{m+k+1}. \tag{20}$$

It should be noted that there results, for each value of n, an infinite set of linear equations. The different sets are identical in form, however, and so it will be necessary to demonstrate the possibility of solution for only one of the sets. For simplicity, the set (16), for which $n=0$, is chosen, and it will be shown by an argument due to von Koch[4] that this system of equations has a unique bounded solution for the A_{0k}. In the argument, α_m will be replaced by its minimum value unity. If the argument holds for this value of α_m, it will hold for all possible values of α_m. (It may be noted that the dependence of α_m upon m is only slight.) Eqs. (16) may then be written:

$$a_{1m}A_{01} + a_{2m}A_{02} + \cdots + (1 + a_{mm})A_{0m} \cdots = K_m\left(\frac{a}{\rho_0}\right)^m \tag{21}$$

where,

$$(m = 1, 2 \cdots \infty).$$

In order to change to von Koch's notation, replace the unknowns $A_{01} \cdots A_{0k}$ by $x_1 \cdots x_k$, etc. and write Eq. (21) as

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + (1 + a_{ii})x_i \cdots = K_i\left(\frac{a}{\rho_0}\right)^i, \tag{22}$$

in which

$$(i = 1, 2 \cdots \infty).$$

Divide (22) by $(1+a_{ii})$ and set

$$\frac{a_{ik}}{1 + a_{ii}} = -b_{ik}, \tag{23}$$

$$b_{ii} = 0,$$

$$\frac{K_i}{1+a_{ii}}\left(\frac{a}{\rho_0}\right)^i = c_i.$$

Then

$$b_{ik} < \frac{(i+k)!}{i!k!}\left(\frac{a}{h}\right)^{i+k+1},$$

$$c_i < K\left(\frac{a}{\rho_0}\right)^i, \tag{24}$$

[4] H. von Koch Jahres bericht der Deutschen Mathematiker Vereinigung **22,** 285–291 (1913).

in which K is the upper limit of K_i, namely D/ρ_0. Eqs. (22) become,

$$x_i - \sum_{k=0}^{\infty} b_{ik}x_k = c_i, \qquad (25)$$

$$(i = 1, 2 \cdots \infty).$$

Let

$$x_k = x_k^{(1)} + c_k,$$

substitute and obtain

$$x_i^{(1)} = \sum_{k=1}^{\infty} b_{ik}x_k^{(1)} + \sum_{k=1}^{\infty} b_{ik}c_k + c_i^{(1)}.$$

Let

$$x_k^{(1)} = x_k^{(2)} + c_k^{(1)},$$

and so on, continuing this process of successive approximations indefinitely. Using the notation

$$b_{ik}^{(2)} = \sum_{v=1}^{\infty} b_{iv}b_{vk}$$

$$b_{ik}^{(3)} = \sum_{v=1}^{\infty} b_{iv}^{(2)}b_{vk}, \qquad \text{etc.},$$

then according to von Koch's argument, the series

$$x_i = c_i + c_i^{(1)} + c_i^{(2)} + \cdots,$$

i.e.

$$x_i = c_i + \sum_{k=1}^{\infty} b_{ik}c_k + \sum_{k=1}^{\infty} b_{ik}^{(2)}c_k + \sum_{k=1}^{\infty} b_{ik}^{(3)}c_k + \cdots, \qquad (26)$$

is absolutely convergent and represents the unique bounded solution of the system (25), provided $\sum_{k=1}^{\infty}|b_{ik}|<1$ for all values of i and provided $c_i<K$ a fixed constant. The first of these conditions is fulfilled, provided $a/h<1/2$, as may be seen from the following summation,

$$\sum_{k=1}^{\infty} b_{ik} = \sum_{k=1}^{\infty} \frac{(i+k)!}{i!k!}\left(\frac{a}{h}\right)^{i+k+1} = \left[\left(\frac{a}{h-a}\right)^{i+1} - \left(\frac{a}{h}\right)^{i+1}\right]. \qquad (27)$$

The second condition is fulfilled, as from (24)

$$c_i < K\left(\frac{a}{\rho_0}\right)^i$$

where

$$\frac{a}{\rho_0} < 1$$

and

$$K \leq \frac{D}{\rho_0}.$$

The first five equations of the system (22) for the special case

$$\frac{a}{h} = \frac{1}{2}, \quad \frac{a}{\rho_0} = \frac{1}{2}, \quad K_i = K,$$

will be written down;

$$\left.\begin{aligned}
1.125x_1 + 0.187x_2 + .125x_3 + .078x_4 + .047x_5 \cdots &= \tfrac{1}{2}K \\
.187x_1 + 1.187x_2 + .156x_3 + .117x_4 + .082x_5 \cdots &= \tfrac{1}{4}K \\
.125x_1 + .156x_2 + 1.156x_3 + .136x_4 + .110x_5 \cdots &= 1/8K \\
.078x_1 + .117x_2 + .136x_3 + 1.136x_4 + .123x_5 \cdots &= 1/16K \\
.047x_1 + .082x_2 + .109x_3 + .123x_4 + 1.123x_5 \cdots &= 1/32K
\end{aligned}\right\} \quad (28)$$

The calculated values of x_1 and x_2, using the first four terms of formula (26), are

$$x_1 = .3716K,$$
$$x_2 = .1456K.$$

It is very laborious, however, to use formula (26) and it has been found that the values of the unknowns x_k may be found more easily by solving the first of the Eqs. (28) for x_1, retaining only the first term; using this value in the second equation to obtain an approximate value of x_2 by retaining the first two terms only; then correcting the value of x_1, by substituting the value found for x_2 in the first equation and so on. By this method, using three of the equations in three of the unknowns, it is found that

$$x_1 = .3725K,$$
$$x_2 = .1458K,$$
$$x_3 = .0475K,$$

in good agreement with the values as calculated by the formula (26). It should be noted that the values of the coefficients fall off rapidly in the present case. If the ratio of $a/h < 1/2$ or $a/\rho_0 < 1/2$, the convergence will be still more marked.

All that has been said in regard to Eqs. (16) may be said of the infinite set of Eqs. (17). The convergence, as n grows, becomes more rapid. For n greater than five, the A_{nk} and B_{nk} can be found to an accuracy of about two percent by making the approximation in (16) and (17)

$$A_{nm} = \frac{1}{\alpha_m} K_{nm} \left(\frac{a}{\rho_0}\right)^m,$$

$$B_{nm} = \frac{1}{\alpha_m} K_{nm} \left(\frac{a}{\rho_0}\right)^m.$$

For n less than five, and $a/h = 1/2$, the values of the coefficients A_{nk} and B_{nk} can be found with an accuracy of the order of one percent by solving the first five equations, in the first five unknowns, of the infinite sets of Eqs. (16) and (17).

The expression for the total potential, due to the buried sphere and the electrode, at all points without the buried sphere is

$$\Phi = \sum_{m=0}^{\infty} \left(\frac{a}{\rho}\right)^{m+1} Y_m(\mu, \phi) + \sum_{m=0}^{\infty} \left(\frac{a}{\rho'}\right)^{m+1} Y_m(\mu', \phi) + \frac{D}{\rho_0} \sum_{m=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^m L_m(\mu_0, \phi_0, \mu, \phi), \tag{29}$$

in which the coefficients in each of the spherical harmonics $Y_m(\mu, \phi)$ and $Y_m(\mu', \phi)$ have the values found by solving the sets of linear Eqs. (16) and (17) for given values of a/h and a/ρ_0.

The first and third members of Eq (29). are expressed in terms of ρ, θ and ϕ. The second member is expressed in terms of ρ', θ' and ϕ. If actually there existed the two spheres S and S_i, then the first and second members of Eq. (29) would express the potential due to the spheres S and S_i respectively. If, as is the case here, there is only the buried sphere S in the infinite half space, which is bounded by the plane MN, then the first two members of (29) are necessary for expressing the potential due to the buried sphere alone. The second member of (29) simply serves to accomodate the boundary condition expressed by Eq. (5).

Numerical Computation

As an example, the potential $(\Phi - U_c)$ will be computed for a point on the surface of the half space directly above the center of the sphere. The potential will be computed for various depths of the sphere and for the ratio of conductivity, k, of the two media equal to ten. The calculation will be made first by means of approximation formulae described below, and finally by means of the exact formula (29). An electrode of radius a_1, at which a current I is being introduced, is placed at the point whose cartesian coordinates are $(h/2, 0, (3)^{\frac{1}{2}}a)$. The potential is calculated for the point whose cartesian coordinates are $(h/2, 0, 0)$. It will be assumed that the current is removed at some electrode so far distant from the buried sphere, that its influence upon the sphere may be neglected. It should be mentioned, however, that if the second electrode were not so situated, the potential of the buried sphere, under the influence of this second electrode, could be computed by means of the formula (29) and this potential added to the potential calculated for the first electrode.

First approximation. As a first order approximation to the potential, the field due to the electrode is considered uniform over the buried sphere and

equal to its value at the center of the sphere, and the interaction between the buried and image sphere is entirely neglected. The potential at all points without the buried sphere then is given by the expression

$$\Phi - U_e = \left(\frac{a}{\rho}\right)^2 [A_1 \cos\theta + (A_{11}\cos\phi + B_{11}\sin\phi)\sin\theta] + \left(\frac{a}{\rho'}\right)^2 [A_1 \cos\theta' + (A_{11}\cos\phi + B_{11}\sin\phi)\sin\theta'], \tag{30}$$

in which the coefficients have the values

$$A_1 = \frac{D}{\rho_0}\frac{(1-k)}{(k+2)}\frac{a}{\rho_0}\cos\theta_0,$$

and

$$\left.\begin{matrix} A_{11} \\ B_{11} \end{matrix}\right\} = \frac{D}{\rho_0}\frac{(1-k)}{(k+2)}\frac{a}{\rho_0}\left\{\begin{matrix} \cos\phi_0 \\ \sin\phi_0 \end{matrix}\right\}\sin\theta_0.$$

Second approximation. As a second order approximation to the potential, the field over the buried sphere due to the electrode will again be considered uniform but the first order cross effect due to the interaction of the two spheres will be taken into account. By first order cross effect is meant that the field, due to sphere S_i, is considered uniform over the buried sphere S and equal to the value it has at the center of the sphere S. Such a formula can be obtained by applying boundary condition (2) and then dropping all terms containing $(a/\rho_0)^2$ and $(a/h)^4$ and higher powers of these ratios. If this be carried out, the resulting expression for the potential is identical with Eq. (30), but the coefficients have the values:

$$A_1 = \frac{D}{\rho_0}\frac{(1-k)}{[(k+2)+(a/h)^3(k-1)]}\frac{a}{\rho_0}\cos\theta_0,$$

and

$$\left.\begin{matrix} A_{11} \\ B_{11} \end{matrix}\right\} = \frac{D}{\rho_0}\frac{(1-k)}{[(k+2)+(a/h)^3(k-1)]}\frac{a}{\rho_0}\left\{\begin{matrix} \cos\phi_0 \\ \sin\phi_0 \end{matrix}\right\}\sin\theta_0.$$

Third approximation. As a third order approximation, the field over the buried sphere due to the electrode will be considered uniform so that terms in $(a/\rho_0)^2$ and higher powers may be dropped. However, the field due to sphere S_i, will no longer be considered uniform over sphere S. That is, terms in $(a/h)^4$ and higher powers will be retained. The formula found under these conditions is:

$$\Phi - U_e = \sum_{m=0}^{\infty}\left(\frac{a}{\rho}\right)^{m+1} Y_m(\mu,\phi) + \sum_{m=0}^{\infty}\left(\frac{a}{\rho'}\right)^{m+1} Y_m(\mu',\phi),$$

in which the coefficients must be determined from the infinite sets of equations of the type (22). Since terms in $(a/\rho_0)^2$ are to be neglected, the right sides of Eqs. (22) after the first are to be set equal to zero.

The values of the potential $(\Phi - U_e)$ as calculated by these three approximate formulae and finally from the exact formula (29) are shown in Fig. 2.

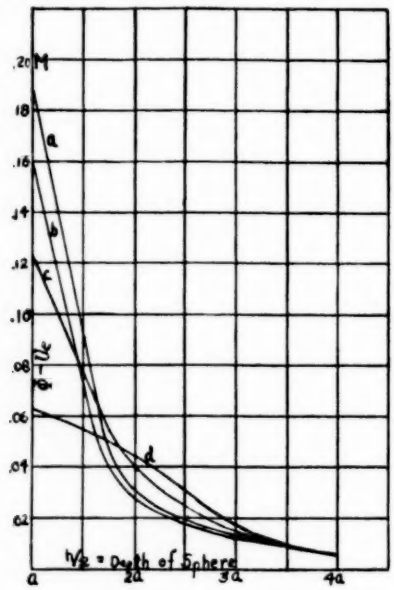

Fig. 2. Potential curves. (a) first approximation; (b) second approximation; (c) third approximation; (d) actual.

To obtain the total potential Φ simply add to the values on the curve, the constant term due to the electrode,

$$U_e = I/4\pi\sigma_2(3)^{1/2}a = M/(3)^{1/2}.$$

This last equation serves to define M, which occurs in the data. In order that the data may be readily put into practical units, the following calculation is to be noted. The potential due to the electrode is

$$U_e = D/r,$$

where r is the distance from the electrode. If, in particular, $r = a_1$, a_1 being the radius of the electrode,

$$U_e = I/4\pi\sigma_2 a_1$$

or

$$I = (U_e)_{a_1} 4\pi\sigma_2 a_1.$$

Therefore, M in the above data becomes

$$M = a_1/a(U_e)_{a_1},$$

and since $(U_e)_{a_1}$ is the potential of the electrode in volts, M will also be in volts.

From the curves, it is clear that any of the approximate formulae are sufficiently accurate for computing the potential, due to the buried sphere, in cases where the depth of the center of the sphere is greater than three times the radius of the sphere. In the region for which $h/2$ is less than a, only the exact formula can be used with any degree of accuracy.

In conclusion, I should like to express my sincere thanks to Professor Warren Weaver of the University of Wisconsin for his helpfulness and interest in the above problem.

MAGNETIC SUSCEPTIBILITIES AND IONIC MOMENTS IN THE PALLADIUM AND PLATINUM GROUPS

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(Received December 15, 1930)

Abstract

Susceptibility measurements have been made on Ru, Rh, Pd, Os, Ir, and Pt and some of their compounds over a range of temperatures by means of an improved Curie balance.

Pd and Pt are found to follow approximately a Curie-Weiss law. Ru, Rh, Os, and Ir are found to have paramagnetic susceptibilities which increase with temperature.

$RuCl_3$ and IrO_2 are found to follow a Curie-Weiss law which gives for the ion Ru^3 a moment of 9.8 Weiss magnetons per mole and for Ir^4 13.8. The spectroscopically predicted value for each ion is 29.4. The paramagnetic susceptibility of Rh_2O_3 is found independent of temperature while that of RuO_2 increases rapidly with temperature. $RhCl_3$ is found to have an unmeasureably small susceptibility over the temperature range investigated. $IrCl_3$ and $PtCl_2$ are diamagnetic with susceptibilities independent of temperature.

THE determination of ionic magnetic moments in the Pd and Pt groups, by susceptibility measurements on compounds of these elements, is of particular interest in connection with the Hund method[1] of predicting these moments from spectroscopic theory. Hund[2] has employed his method for the moments of the trivalent rare earth ions and obtained remarkable agreement with the measured values of several investigators. In the case of the ions of the iron group, the elements Sc to Ni of the periodic table, the method has been unsuccessful. Laporte and Sommerfeld[3] calculated the moments to be expected for the iron group ions in two cases, infinite separation and zero separation of the J-levels of the ground multiplet predicted by the Hund method. Several of the measured values are found to lie outside of these limits. Bose[4] has pointed out that the measured moments of the iron group ions are best explained by assuming that they are due to the spin of the electrons alone, the orbital moments being ineffective. Stoner[5] has called attention to the striking difference in the structures of the rare earth and iron group ions, and the effect that this difference should be expected to have on the ions with the conditions under which their moments are measured. In the iron group ions the incompleted group of electrons which gives the ion its mechanical and magentic moments is that of highest total quantum number. In the rare

[1] F. Hund, Zeits. f. Physik **33,** 345 (1925).
[2] F. Hund, Zeits. f. Physik **33,** 855 (1925).
[3] O. Laporte and A. Sommerfeld, Zeits. f. Physik **40,** 333 (1926).
[4] D. M. Bose, Zeits. f. Physik **43,** 864 (1927).
[5] E. C. Stoner, Phil. Mag. **8,** 250 (1929).

earth ions this group of electrons is not that of highest total quantum number. The interaction between the paramagnetic ion and surrounding ions and molecules will presumably effect primarily the electrons in the group of highest total quantum number; in the iron group ions the electrons responsible for the magnetic moment, in the rare earth ions electrons that are not responsible for the magnetic moment. In general this interaction is a strong orbital interaction while the spin moments are left relatively free. In a qualitative way this is an explanation of the apparent ineffectiveness of the orbital moments in the iron group ions and their almost complete effectiveness in the rare earth ions.

There exist few data on the susceptibilities of the elements of the Pd and Pt groups from which ionic moments can be calculated. Bose and Bhar[6] have made measurements at room temperature on some chlorides and complex salts of some of these elements and have calculated ionic magnetic moments assuming that the susceptibilities of the compounds follow the Curie law. Cabrera and Duperier[7] made measurements on $RuCl_3$, $RhCl_3$, $PdCl_2$, $OsCl_2$, $IrCl_3$, and $PtCl_2$ over a range of temperatures from about 20°C to 100°C. They found that $RuCl_3$ approximately followed a Weiss law which gave a moment for Ru^3 of 12.9 Weiss magnetons. The susceptibilities of the other chlorides were found to remain constant or increase with temperature.

For this investigation the compounds $RuCl_3$, RuO_2, $RhCl_3$, Rh_2O_3, $IrCl_3$, IrO_2, and $PtCl_2$ were obtained from the American Platinum Works, Newark, N.J. These compounds were prepared with special regard for purity by Dr. S. Streicher of their chemical laboratory.

Through the kindness of the Bureau of Standards samples of the six metals Ru, Rh, Pd, Os, Ir, and Pt of very great purity were made available for a new determination of their susceptibilities. Several investigators have measured the susceptibilities of these metals but the results have been in very poor agreement.

Description of Apparatus

The susceptibility measurements were made by means of a Curie balance. In this apparatus the usual mechanically operated torsion head was replaced by an electrically operated one. The ease and speed of making measurements with this arrangement far surpasses that attainable with the usual mechanical types. In Fig. 1 is shown a sketch of the essential parts of the electrically operated torsion head. A coil of small copper wire, A, wound on a flat, hard rubber spool is suspended by a fine wire, B, with its plane parallel to a magnetic field produced by two fixed coils, C, carrying a current. To the coil A is attached the suspension, D, which supports the balance arm and to which is applied the torque to overcome the force on the sample. A plane mirror, E, is attached to the coil A and the twist in the suspension D is read from it by means of a telescope and scale placed about one meter away. In order to fix the axis about which the coil A rotates suspension guides, F, are em-

[6] D. M. Bose and H. G. Bhar, Zeits. f. Physik **48**, 716 (1928).
[7] B. Cabrera and A. Duperier, C. R. **185**, 414 (1927).

ployed. These guides consist of thin brass strips attached to the box which supports the balance and pierced with small holes through which the suspensions pass. The coil *A* is damped by means of vanes *G* suspended in stanolax. A current, which is controlled so as to give the proper torque to the suspension, is carried through the coil *A* by means of long thin spirals of wire which do not hinder the coils free rotation. The other parts of the balance are of the usual design.[7a]

The apparatus was calibrated by making measurements on distilled water, using as its susceptibility with respect to air at 20°C and 76 cm pressure the value -0.749×10^{-6}.

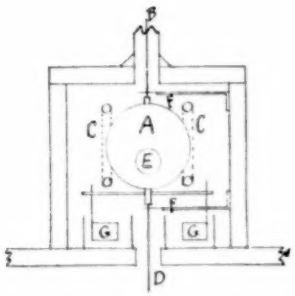

Fig. 1. Sketch of the essential parts of the electrically operated torsion head.

The inhomogeneous magnetic field was furnished by a Dubois electromagnet with its pole pieces set at an angle of about twelve degrees with a line joining their centers. The field strengths obtainable were in the range 1400 to 6400 gausses.

The samples were heated by means of a nichrome resistance furnace. A junction of a Cu-constantan thermocouple was mounted so as to occupy the same relative position in the furnace as the sample on which measurements were being made. The other junction of the thermocouple was kept in a bath of ice and water. Temperatures were read by means of a D'Arsonval galvanometer in the thermocouple circuit. A galvanometer deflection of one mm corresponded to a temperature change of about 2.5°C. The thermocouple was calibrated by means of the boiling points of well-known substances.

Experimental Results

In Tables I, II, III, are given the results of measurements on the various substances indicated.

The susceptibility of Rh was found to decrease with increasing field strengths. At room temperature the change was from 1.60×10^{-6} at 1420 gausses to 1.20×10^{-6} at 6390 gausses. This variation was assumed to be due to ferromagnetic impurities and the method first used by Honda[8] was used to calculate the susceptibility of Rh from these measurements.

[7a] The modification of the Curie balance as used by Foëx was considered but calculation showed that the torsion method could be made more sensitive for our measurements.

[8] K. Honda, Ann. d. Physik (4) **32,** 1027 (1910).

TABLE I.

$T°K$	Ru	Rh*	Pd	$\chi\times10^6$ RuO_2	Rh_2O_3
298	0.427	1.08	5.15	1.21	0.408
333	.431	1.09	4.79	1.24	.409
380	.435	1.11	4.39	1.28	.410
433	.443	1.12	4.03	1.32	.403
480	.452	1.14	3.73	1.36	.403
523	.457	1.15	3.52	1.40	.402
573	.466	1.16	3.27	1.43	.401
623	.475	1.17	3.05	1.46	.398
673	.487	1.18	2.85	1.49	.397
723	.496	1.19	2.66	—	—

* Calculated from the measurements by the method of Honda.

TABLE II.

$T°K$	Os	Ir	$\chi\times10^6$ Pt	IrO_2
298	0.052	0.133	0.982	0.997
348	—	.138	.947	.984
398	.059	.141	.925	.974
473	—	.146	.876	.971
548	.065	.151	.831	.942
623	—	.159	.795	.926
698	.070	.167	.745	.891

TABLE III.

$T°K$	$RuCl_3$	$RhCl_3$	$\chi\times10^6$ $IrCl_3$	$PtCl_2$
298	7.21	0.00	−0.114	−0.151
323	6.64	.00	—	—
348	6.16	.00	−0.114	−0.149
373	5.77	.00	—	—
398	5.44	.00	—	−0.149
423	5.18	.00	—	—
448	4.91	.00	—	—

For comparison the available data on the specific susceptibilities of the metals Pd, Rh, Ru and Pt at 18°C and the investigators who obtained them are given in Table IV.

TABLE IV.

Investigator	Pd	$\chi\times10^6$ Rh	Ru	Pt
Curie	5.2	—	—	—
Honda	5.8	1.14	0.56	1.10
Owen	5.2	1.08	0.43	0.90
Onnes and Oosterhuis	5.3	—	—	—
Foëx	5.27	—	—	1.028
Kopp	5.29	—	—	—
Authors	5.24	1.08	0.426	1.015

In Fig. 2 are plotted the values of $1/\chi$ for Pd, Pt, $RuCl_3$ and IrO_2 as a function of temperature.

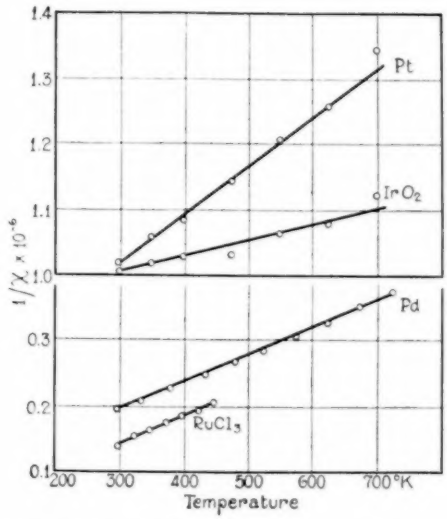

Fig. 2. Values of $1/\chi$ for Pd, Pt, $RuCl_3$ and IrO_2 plotted as a function of temperature.

Discussion of Results

Pd and Pt are found to follow approximately a Curie-Weiss law, the deviation being accounted for by an underlying diamagnetism independent of temperature. From purely formal theory, it would be expected that the diamagnetic component of the susceptibility would be greater for Pt than for Pd but the experimental results indicate the opposite.

Ru, Rh, Os, and Ir are found to have paramagnetic susceptibilities which increase with temperature. This is, of course, inconsistent with Langevin's theory.

$RuCl_3$ and IrO_2 are found to follow a Curie-Weiss law. For the former the equation is:

$$\chi = \frac{2340}{T + 37} \times 10^{-6} \text{ c.g.s./gram}$$

and for the latter

$$\chi = \frac{4310}{T + 4025} \times 10^{-6} \text{ c.g.s./gram.}$$

These results give for the magnetic moment of the Ru^3 ion 9.8 Weiss magnetons per mole, and for the Ir^4 ion 13.8. The predicted normal state for both ions is ^6S which gives 29.4 Weiss magnetons per mole for the moments. In the iron group, good agreement between measured and predicted moments has been found for all ions with predicted S normal states. In these groups, there is no agreement whatever and the behavior of some of the compounds is such

that no moments can be determined from measurements on them. If the susceptibility does not follow a Curie-Weiss law, there is no theory which will give the elementary magnetic moments.

A deviation from the Curie-Weiss law is predicted for compounds containing a paramagnetic ion with other than an S normal state due to the increase with temperature of the number of excited ions with moments differing from that of the ions in their normal state. With an S normal state there are no excited ions at ordinary temperatures. In these researches only the compounds containing the paramagnetic ion with a predicted S normal state are found to follow a Curie-Weiss law but the deviations of the other compounds are opposite to (excepting RuO_2) and much greater than can be accounted for by the theory.

The behavior of those compounds which do not follow a Curie-Weiss law is not due to a permanent chemical or physical change due to heating for at room temperature the susceptibilities are found to be the same after as before heating.

The disagreement between the measured and the spectroscopically predicted ionic moment may be due to error in prediction of the normal state. Hund's rule, combined with present ideas of the electronic arrangement in atoms and ions, predicts for Ru^3 and Ir^4 a ^6S normal state. In the same way the prediction for the normal state of the Pd atom is a ^3F. The atomic spectrum of Pd has been analysed by Beals, Beckert and Catalan, and McLennan and Smith.[9] They all have concluded that the normal state for the Pd atom is a ^1S which means that the Pd atom in its normal state has zero magnetic moment. That the moment of the Pd atom is zero in its normal state has been verified by Copley and Guthrie[10] by the molecular ray method.

The question of the validity of Hund's rule in the wave mechanics has been examined by Slater.[11] He has found that the wave mechanics gives the same normal state as Hund's rule although their predictions differ higher up in the spectrum.

The interpretation of the magnetic moment determined from susceptibility measurements on compounds as the true moment of the paramagnetic ion seems not to be well founded. The ion in chemical combination, surrounded by other ions and molecules, must be a different thing from the ion with which the spectroscopist deals in the Zeeman effect. Therefore, in spite of some past successes it appears hopeless to expect always to find agreement between these two.

The writers wish to express their appreciation to Professors E. H. Williams and Jakob Kunz for their encouragement and advice during the progress of this work.

[9] See F. Hund, Linienspektren und Periodische System der Elemente (1927).

[10] Not yet published.

[11] J. C. Slater, Phys. Rev. **34,** 1293 (1929).

CHANGES IN ELECTRICAL RESISTANCE DUE TO MAGNETISM AND HARDNESS

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(Received December 19, 1930)

Abstract

A series of twenty-two rods or strips of nickel, to which had been imparted different degrees of hardness by cold rolling, had their resistance measured when subjected to a longitudinal magnetic field. There were eleven degrees of hardness, two samples for each degree. The longitudinal magnetic field increased the resistance in all cases.

The different degrees of hardness did not vary the specific resistance of the nickel, but it did cause a large variation in the change of resistance due to a magnetic field. The measurements were carried out at a constant temperature of 35°C.

IN HIS summary to Part I of an extensive study of the electrical conductivity in strong magnetic fields, Kapitza[1] says, "It has been shown that the physical change produced in a conductor by hardening and annealing has a strong influence on the phenomenon of change of resistance. The influence of the impurities is also very marked and was studied."

Kapitza has brought together as fine a collection of the elements for study in strong magnetic fields as has ever been assembled, but his greatest difficulty was in procuring these samples in a pure state. It has been pointed out[2] that very worthwhile results may be obtained by comparing magnetic phenomena in the same specimen, even though the purity of the sample may not be vouched for.

Some years ago there was begun in this laboratory a study[3] of a series of nickel rods or strips, twenty-two samples in all, with eleven degrees of hardness in the series. The rods were given different degrees of hardness by cold rolling to various degrees of reduction. The rods were all of the same original heat and had the same chemical analysis, which is herewith given: copper, 0.16%; nickel, 98.8%; iron, 0.56%; sulphur, 0.008%; silicon, 0.06%; carbon, 0.09%; manganese, 0.23%.

While these rods could not be considered as pure nickel, yet they could be said to be identical specimens so far as their chemical constitution was concerned. It was, therefore, possible to study the effect of different degrees of hardness on the electrical conductivity of a definite material when magnetized longitudinally.

The rods came in flat strips and were furnished through the courtesy of the International Nickel Company. There were, originally, eleven strips, 91

[1] Kapitza, Proc. Roy. Soc. **119,** 358 (1928); **123,** 292 (1929); **123,** 342 (1929).
[2] Williams, Phys. Rev. **34,** 258 (1912); **4,** 498 (1914).
[3] Williams, Trans. A. S. S. T. p. 885, 1926; Science **65,** p. 306 (1927); **66,** p. 358 (1927).

cm long, 4.8 cm wide and 0.61 cm thick, all annealed to the same degree. Ten of these were then given the percentage cold reduction shown in Table I.

TABLE I.

Strip number	1	2	3	4	5	6	7	8	9	10	11
Percentage reduction	0	9.7	18.9	28.9	29.5	50.0	59.5	69.0	79.0	89.1	93.3

Each one of these eleven strips was then cut in two, longitudinally, and given a common length of 58 cm. This made a total of twenty-two specimens in all. The cross-sectional dimensions of the completed strips are given in Table II. They are the averages of ten to twelve readings taken along the length of the rods with a micrometer caliper. The two sets of strips were numbered;

$$A_1, A_2, A_3, \cdots, A_{11}$$

and

$$B_1, B_2, B_3, \cdots, B_{11}$$

wherein the rods A and B with the same subscript formed one of the eleven original strips.

TABLE II.

Strip number	Cross-section of strips	
	Series A	Series B
1	0.953×0.601	0.952×0.603
2	.956× .552	.955× .549
3	.955× .496	.952× .495
4	.942× .435	.944× .434
5	.945× .372	.943× .370
6	.956× .306	.964× .306
7	.960× .248	.961× .249
8	.956× .193	.956× .193
9	.956× .131	.957× .131
10	.957× .069	.956× .069
11	.955× .043	.956× .044

The scleroscope hardness, (universal tup or hammer) was furnished by the International Nickel Company and is given in Table III.

TABLE III.

Strip number	1	2	3	4	5	6	7	8	9	10	11
Scleroscope hardness	17	27	31	35	40	43	45	47	50	51	49

The resistance was measured between two fiducial marks 51.8 cm apart on each rod. The measurements were made by means of a Kelvin double bridge. The complete outfit was one manufactured by Leeds and Northrup Company and found to be most satisfactory. The bridge was made sensitive enough to indicate easily a change of resistance less than 0.000001 ohm. Higher sensitivity created more difficulty in making observations.

In Fig. 1 is shown the arrangement of apparatus. The longitudinal magnetic field was obtained by means of a solenoid, S, one meter long. The internal diameter of free space was 5.5 cm. There were 6025 turns which gave a constant of 75.3 gausses per ampere, both by calculation and by experimental test. In order to keep the strips at a constant temperature while being tested, they were placed in an oil bath. The container for the oil consisted of a copper trough, TT, passing through the solenoid and insulated from it by a coating of shellaced cloth. By means of brass gas-pipe fittings this trough was connected with a rectangle G, of $\frac{3}{4}$ inch brass pipe. The pump, P, kept the oil in continuous circulation around the rectangle. This pump consisted of a propeller blade inside of a tube and driven by the small motor, M. Transformer oil was used for circulating in the trough. Inasmuch as heat was

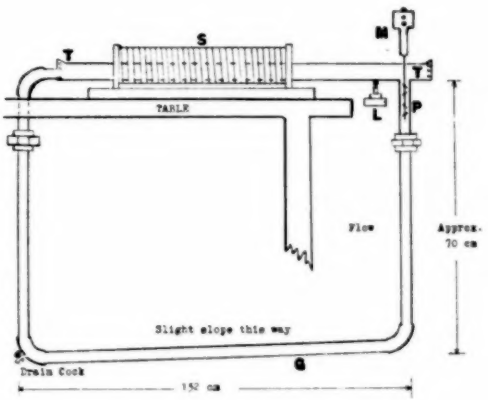

Fig. 1. Arrangement of magnetizing solenoids and constant temperature bath.

slowly given up to the system by the solenoid, it seemed best to keep the temperature of the bath somewhat above that of the room. A temperature of 35° C, was chosen as a suitable point from which the bath would always be cooling off. The moment it got below 35°C, the thermostat would turn on more gas to the burner, located under one end of trough, and keep the temperature up at 35°C. The bath did not vary over 0.2 degrees during the measurements. This kept the temperature sufficiently constant for making comparable resistance measurements. The temperature of the bath was determined by a sensitive, calibrated mercury-in-glass thermometer. Sometimes it was found that manual control of the gas was helpful. The thermostat for controlling the gas supply at the burner, L, which heated the bath, was a simple mercury-in-glass device. All the values for resistances, given in this paper, are for the temperature of 35° C, unless otherwise stated.

As found by other investigators,[4] the resistance of nickel is increased when magnetized longitudinally. This was true for all of the twenty-two specimens examined. The effect of hardening may best be studied in Fig. 2, where for

[4] W. E. Williams, Phil. Mag. **4,** 430 (1902); **9,** 77 (1905).

the field strength of 75.3 gausses the changes in resistance per ohm is plotted against hardness numbers. The effect of increasing the hardness is to decrease the change in resistance produced by the magnetic field. Fig. 2 is also

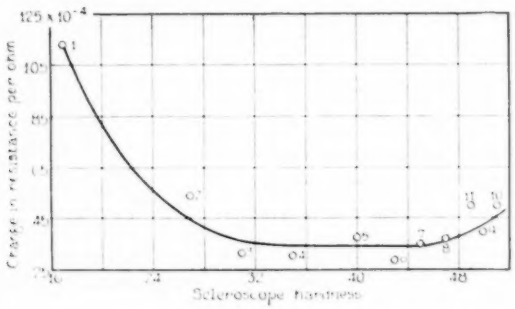

Fig. 2. Change in resistance per ohm for the different nickel strips at a field strength of 75.3 gausses.

of interest in comparison with a curve published in a paper[5] presenting the relation between the change in length due to a magnetic field and the hardness of this same set of strips. This curve is shown in Fig. 3 with a field strength of 58.8 gausses.

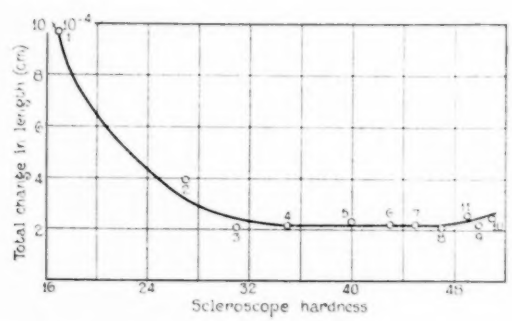

Fig. 3. The changes in length for the different nickel strips at a field strength of 58.8 gausses.

The most rapid change in length and in resistance comes in the first few reductions. In some way or another the changes in resistance due to a magnetic field are very closely related to the changes in length due to a magnetic field. There seems to be a common mechanism operating in both effects.

In Fig. 4 is shown the way in which the change in resistance per ohm varies with increasing magnetic field. Curve 1 is the average of the values for the two rods, A_1, and B_1, while curve 2 is the average for the two rods, A_4 and B_4. This figure represents the greatest difference obtained in the series of strips, due to differences in hardening. It confirms in a very marked way the results obtained by Kapitza, i.e., that hardness plays a very important rôle in this effect.

[5] Williams, Trans. A. S. S. T. p. 885 (1926).

The various values of dr/R in the curves represent the average of at least eight observations. Demagnetization of the rods occurred between each reading. This was accomplished by sending a decreasing a.c. through the solenoid.

In getting the changes in resistance due to magnetism and hardness, it was also easy to find the specific resistance of these strips. The values obtained are given in Table IV, and which are the averages of the *A* and *B* series. Three

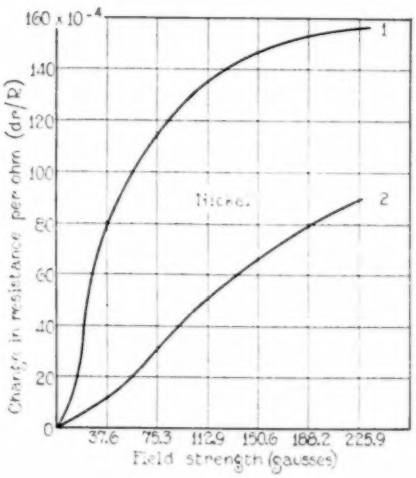

Fig. 4. Change in resistance per ohm as magnetizing force increases.

sets of values are given which were taken by different observers using different methods and about two years apart. The first set is by Mr. Lusk of the Rawson Electrical Instrument Company. Mr. Lusk checked his results by two entirely different methods. The second set is by Mr. G. K. Schoepfle, a former graduate student in this department. The third set is furnished by the junior author of this paper.

This set of readings seemed rather astonishing at first because as one crushes the nickel strips by rolling, and therefore the crystals comprising

TABLE IV.

Strip number	Specific resistance		
	Lusk	Schoepfle	Sanderson
1	0.00001012	0.00001083	0.00001099
2	.00000993	.00001081	.00001102
3	.00000993	.00001035	.00001093
4	.00000995	.00001085	.00001089
5	.00001000	.00001060	.00001086
6	.00000995	.00001071	.00001082
7	.00001000	.00001055	.00001079
8	.00001008	.00001070	.00001071
9	.00001032	.00001090	.00001076
10	.00001033	.00001051	.00001079
11	.00001038	.00001102	.00001073

the strips, it would suggest the possibility that the specific resistance would also be changed in a corresponding manner. *The specific resistance appears to be constant.*

The values given by Lusk were obtained about five years ago and at room temperature. The values by Schoepfle were obtained about two years later and also at room temperature. In all probability there had been some aging process going on in the meantime. To release local strains set up in the rods at the time they were cut and machined into shape, they were annealed for one hour in an oil bath at 304°C. This process took place about seven years before the readings were taken for this paper.

It was also possible with this equipment to determine the temperature coefficient of the strips. This was not worked out completely, but it was carefully measured on the rods, A_1 and A_4, those rods which had the widest variation of dr/R for a given magnetic field Measurements were made at 35°C and at 85°C. This gave A_1 a temperature coefficient of 0.0042 and A_4 the value of 0.0043, indicating that there was little or no change in the temperature coefficient due to difference in rolling.

Although hardening by cold rolling did not change the specific resistance of this series of nickel strip, yet it did affect the change in resistance due to a magnetic field. This seems to be the most significant fact emerging from the results of this investigation,—no change in specific resistance due to varying hardness, but a decided change in the variation of resistance with magnetic field.

Attention should be called to the excellent paper by McKenhan[6] which has appeared since these results were obtained.

[6] McKeehan, Phys. Rev. **36,** 948 (1930).

THE ROLES OF DISCRETE AND CONTINUOUS THEORIES IN PHYSICS*

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Abstract

The differential character of the principal equations of physics implies that physical systems are governed by laws which operate with a precision beyond the limits of verification by experiment. This appears undesirable, from an axiomatic standpoint.

Postulates of real, finite and rational description.—We may safely assume that all results obtained experimentally can be described by using real and finite functions of real, finite and rational variables. It appears desirable, therefore, to use the calculus of finite differences in formulating the results of our experiments. In dynamics, the use of this calculus yields terms corresponding to the fact that the configuration is known only when we observe it and must be arbitrary at intermediate times. For example, if a "uniformly moving particle" is observed once a second, we find experimentally that

$$\Delta^2 x/\Delta t^2 = 0,$$

whence

$$x = Ft + G.$$

Here F and G are *arbitrary* Fourier series having the period Δt. The general difference equations of dynamics are formulated. The Lorentz transformation is reinterpreted, relativistic difference equations of motion and of light propagation are set up, and geodesics in a discrete manifold are discussed.

Extensions of these ideas to electrodynamics are outlined.

I. Introductory

IT IS often said that while the fundamental equations of physics must be capable of describing every feature of our experimental results, at the same time they must not introduce extraneous or unobservable features. On the basis of this criterion, many branches of physics are not in a satisfactory position. Broadly speaking, the difficulties to which we refer originate in the use of mathematical concepts which are unsuited to the description of our experimental knowledge, and which introduce complications having no counterparts in nature.

Let us begin with an example. Consider, on the one hand, the information we obtain by performing an experiment once, or at most, only a few times. Usually, this information consists of a table of related numerical data. Consider, on the other hand, the statistical knowledge we obtain by correlating the results of a very large number of similar experiments. We draw curves and calculate averages, and finally, we may attempt to represent our results in an idealized fashion, by writing empirical equations. Up to this point the

* A preliminary account was given before the American Physical Society in February 1929. See Phys. Rev. **33**, 637 (1929).

procedure is beyond criticism, if we keep in mind the experimental basis of our equations, but when we carry the idealization process one step further, and derive differential equations from the empirical ones, we introduce conceptual difficulties. A differential equation connotes much more than the data from which it arises; herein lies its strength and,—unfortunately,— its weakness. For instance, Newton's equations imply that a dynamical system is governed by laws operating with a precision beyond the limits of detection by experiment. If, at a time t, we know the position and velocity of a body, these equations predict its condition at the time $t+dt$, but in order to verify the prediction, one would be obliged to distinguish between two positions, x and $x+dx$, by making two measurements separated by an infinitesimal time interval. But as a matter of fact, we have no desire to find the condition after the time dt; we wish to predict the state, within limits, after a finite time interval, and this we do by integrating the equations of motion. Each coordinate may then be expressed as a continuous function of the time, and we may calculate the configuration at *any* time, that is, at an "infinite number of instants." The information yielded by the equations is much in excess of what we can test experimentally. No one will deny the great practical value of Newton's equations, and clearly, the point we wish to emphasize is this: from an aesthetic and logical standpoint, it is desirable to have available a mathematical algorithm which makes it possible for us to summarize precisely the information obtained from a series of measurements, without excess and without deficit. It is our aim to show how this may be accomplished with the aid of the calculus of finite differences.

II. Critique of the Use of Infinitesimal Analysis in Physics

The Principle of Real and Finite Description

Most of the purposes we have in mind may be accomplished by adopting a set of postulates which, for brevity, will be collectively designated as the principle of real and finite description. For our present purpose[1] we may state the postulates as follows:

All results obtained experimentally can be described in terms of real and finite functions of real and finite variables. These functions need only be defined for discrete values of their independent variables. Both the functions and their independent variables need only assume rational values.

As an illustration, we consider a series of experiments on the motion of a single atom, or on a small number of atoms. The principle of real and finite description indicates clearly that our numerical results should not be discussed by means of continuous analysis. But if the series of experiments were to be extended indefinitely, until observations had been made on a great many

[1] For the sake of simplicity, we shall speak as though a measurement consisted in assigning a definite number x_0 to a variable at a definite time, t_0. The real situation is that we know x lies in the range $x_0 \pm x_0$ when t lies in the range $t_0 \pm \Delta t_0$. Thus we may say that x is an "indefinite function" of the "indefinite variable" t. It is easy to devise an algebra for dealing directly with the possible connections between such variables; but these matters will be postponed to a later occasion.

individuals, and over the entire ranges of the independent variables, then we would have the material necessary for building up a continuous, statistical theory,—in fact, a branch of wave mechanics. The situation in regard to the motion of a macroscopic body is quite similar. On the basis of a small amount of experience with its motion, we should not write a continuous law. (The true status of the laws of motion is connected with Ehrenfest's theorem, and is discussed in section V.) In the light of these ideas, the use of continuous analysis in physics can be clearly understood. Continuous laws should be introduced only when we deal with statistical results, obtained from vast numbers of experiments.

While the situation in regard to the use of infinitudes in physics is generally unsatisfactory, an exception must be made in the case of wave mechanics, where it is required that the wave functions shall be finite, continuous and single-valued at all real points of the coordinate space used by Schrödinger. The validity of these conditions has sometimes been questioned on the ground that, where the wave function ψ is zero, $1/\psi$ is infinite and there does not seem to be any reason why we should keep ψ from becoming infinite, rather than $1/\psi$. The writer feels that this criticism is invalid, for the square of the absolute value of ψ is measurable, indirectly at least, and therefore must not become infinite for real, finite and rational values of its variables. How ψ behaves for irrational values of its variables is none of our concern.

Again, it may be objected that in wave mechanics integrals extending over the entire ranges of the coordinates x, y and z (from $-\infty$ to $+\infty$) are constantly used with valuable results. Just as in the case above, this appears undesirable since we should think of the coordinates as measurable quantities, and should therefore limit them to finite values. Since in many problems the ψ-function approaches zero very rapidly for large values of the coordinates, it will usually occur that the values of matrix components will be little affected by restricting the range of integration to exclude infinite values, and the practical consequences of the theory will be unaltered. Even in scattering problems, where sinusoidal wave trains are used to represent incident beams of particles or protons, it would be better physics to speak of these trains as having their origin at a great, but finite, distance from the scattering center. Further, it appears desirable to use wave equations which are modified to take account of general relativity. It appears plausible from astronomical evidence that our four-dimensional space-time is limited in extent, and if this is the case, integrals of the type $\int \psi^* f \psi \, dv$ would naturally be extended only over a finite range. Furthermore, from our present stand-point, it is obvious that we should not treat the electron or the proton as point-singularities at which the electromagnetic and gravitational forces become infinite. Indeed, all such artificial mathematical difficulties are excluded by strict adherence to our postulates.

The situation in regard to the use of imaginaries in physics also requires comment. It is generally recognized that the introduction of $\exp i\omega t$ in vibration problems is merely a mathematical artifice, but many a physicist would feel surprise at the proposal to replace the quantum condition, $pq - qp =$

$h/2\pi i$, by an equation which is free of the imaginary unit. However, from an axiomatic and aesthetic standpoint, imaginaries should be excluded from any discussion of fundamental physical questions, and this can always be done by rationalizing our equations, even when we are dealing with quantities which do not commute. (In the present instance we may use $(pq-qp)^2=-h^2/4\pi^2$ or $(pq-qp)(pq-qp)^*=h^2/4\pi^2$. Then, in solving any specific problem, we get the Heisenberg matrices and their imaginary conjugates. In practice, this doubling of the solution will introduce no difficulties, and may even be advantageous.)

After these explanatory remarks we are ready to apply the principle of real and finite description to dynamics and electromagnetic theory. We shall find that when we apply it in dynamics we are naturally led to equations of finite differences, replacing the differential equations of motion. The solution of these difference equations brings in *arbitrary* functions which reduce to known constant values only at times when the configuration of the system is actually observed. This corresponds to the fact that we are not able to state the configuration of the system, except at the instants of observation. We proceed at once to amplify these statements.

III. The Difference Equations of Dynamics

Let us analyze the information obtained by an experimental study of a body moving along the x-axis. For the moment we pass over difficulties arising from errors of observation and assume that the observer can determine times and distances with any desired precision. The result of a set of position measurements will be a table showing the values of x at each instant of observation, such as the following:

$$\begin{aligned} &t_1, t_2, t_3, \cdots, t_n; \\ &x_1, x_2, x_3, \cdots, x_n. \end{aligned} \tag{1}$$

With a view to simplicity, suppose the intervals between successive observations have a constant duration, Δt. This restriction is unnecessary, and later we shall remove it. Perhaps the most natural method of analyzing the data is to study the difference coefficients of the series of x-values. Writing $x_2-x_1=\Delta x_1$, $\Delta^2 x_1=\Delta x_2-\Delta x_1$, etc., we form the table (2).

$$\begin{array}{lll} x_1 & & \\ & \dfrac{\Delta x_1}{\Delta t} & \\ & & \dfrac{\Delta^2 x_1}{\Delta t^2} \\ x_2 & & \\ & \dfrac{\Delta x_2}{\Delta t} & \\ & & \dfrac{\Delta^2 x_2}{\Delta t^2} \\ x_3 & & \\ & \dfrac{\Delta x_3}{\Delta t} & \end{array} \tag{2}$$

$$x_4$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Let us suppose we find that

$$\frac{\Delta x_i}{\Delta t} = \text{constant}, \quad i = 1, \cdots, n-1, \tag{3}$$

and

$$\frac{\Delta^2 x_i}{\Delta t^2} = 0, \qquad i = 1, \cdots, n-2. \tag{4}$$

From our present standpoint, such a motion will be defined as uniform, and the constant value of $\Delta x_1/\Delta t$ is *defined* as the velocity. Either one of the relations (3) and (4) enables us to construct the original table of x-values,[2] but in order to attain a degree of generality equivalent to that provided by the equation $d^2x/dt^2 = 0$ we shall deal with (4). Taken as they stand, the Eqs. (3) and (4) state a regularity of the data and say nothing about the values of $\Delta x/\Delta t$ and of $\Delta^2 x/\Delta t^2$ at instants which lie outside the period covered by the experiment. Further, they say nothing about the values of $\Delta^2 x/\Delta t^2$ for times intermediate to the instants of observation. They contain nothing more than the isolated facts revealed by the experiment, and we must recognize the element of hypothesis introduced when we use them as a basis for prediction. The most natural method of generalizing (4) is to write

$$\frac{\Delta^2 x}{\Delta t^2} = 0, \tag{5}$$

and to consider this equation, provisionally, as applying to *any* series of observations like the above, obtained at time intervals Δt and extending over a finite time. Now, the general solution of (5) is not identical with the general solution of $d^2x/dt^2 = 0$. In fact, it is

$$x = Ft + G, \tag{6}$$

where F and G are trigonometric series of period Δt,[3] let us say

$$F = A_0 + A_1 \sin 2\pi\frac{t}{\Delta t} + A_2 \sin 4\pi\frac{t}{\Delta t} + \cdots + B_1 \cos 2\pi\frac{t}{\Delta t} + \cdots, \tag{7}$$

the expression for G being similar. The coefficients in these series are arbitrary constants, except for one restriction. We require that F shall reduce to the observed velocity v, and G to the observed initial value of x, let us say x_0, whenever t is a multiple of Δt. The coefficients in (7) are *not* Fourier coefficients which cause the series to represent the function $x_0 + vt$, in the interval

[2] In using equation (3) to reconstruct the x-values, we must specify x at one of the instants of observation, let us say t_1; and in using (4), we must specify one value of x and one of $\Delta x/\Delta t$.

[3] Boole, Finite Differences, reprint by Stechert, p. 80.

from $t=0$ to $t=\Delta t$. On the contrary, when an arbitrary choice of the form assumed by x between observations has been made, they are the Fourier coefficients of that form, and the series will generally be infinite. However, the series may contain only a finite, but very great, number of terms, if we specify the values of x at only a finite number of instants. The latter type of choice is in fact the only type which is in agreement with the principle of real and finite description. With these understandings, Eq. (6) completely reproduces the experimental results. At intermediate times, when the system is not under observation, the arbitrary trigonometric terms come into play, which means that the position of the system is not known.[4] The state of affairs is illustrated by Fig. 1, in which the solid line represents the classical motion, while the dotted curve shows a path which might possibly be represented by (6) and (7). The dotted curve might even have a finite number of disconti-

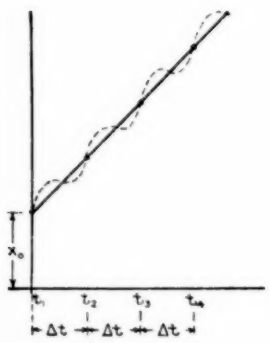

Fig. 1.

nuities in each interval Δt. When the specified set of observations is completed, we have no *physical evidence* which excludes the possibility of such a curve as the dotted one in Fig. 1. Of course, we could have taken additional observations, at the times $\Delta t/2$, $3\Delta t/2$, etc., let us say, obtaining an equation similar to (6). Under these circumstances, however, the state of our knowledge is different, for now the periodic function x must have the same values as the function $vt+x_0$, whenever t is a multiple of one-half the original Δt; at all other times it is arbitrary. The number of observations at times between t_1 and t_n cannot be increased indefinitely; therefore the Eqs. (5), (6) and (7), since they yield the observed data only and nothing more, represent a closer approximation to our experimental knowledge of the motion than the corresponding differential equation and its solution. However,

[4] In a paper entitled "The Limits of Accuracy of Physical Measurements," Proc. Nat. Acad. Sci. **14,** 322 (1928), the writer has shown that there are definite natural limitations on the accuracy with which we can measure a coordinate or specify a time, quite independent of all considerations of experimental skill. If we are dealing with a free particle of mass m, we have $\Delta x_{\min}=h/mc$ and $\Delta t_{\min}=h/mc^2$. It is interesting to note that if the time interval between observations is reduced to the order of $\Delta t_{\min}$, the first trigonometric term in (7) has a frequency of the order mc^2/h; that is, the frequency of the matter waves for a particle of mass m.

(6) still suffers from several defects: (1) it definitely prescribes the value of x at instants lying outside the range t_1 to t_n, although the observations were made only within that range; (2) it is scarcely arbitrary enough at intermediate times, for the trigonometric terms represent a periodic function; (3) it is assumed that the observations are equally spaced. In removing these defects, we may as well speak of a general system, exposed to forces of any kind whatever. Let all the coordinates, $q_1, \cdots q_r$ be observed at the instants $t_1, t_2, \cdots t_n$. We desire to construct a set of functions $q_1(t), q_2(t), \cdots$ which will reduce, at the observation times, to functions which adequately represent the observed values of the coordinates and which will be arbitrary at all other times. The solution is immediate.[5]

We need only write down the polynomials obtained by applying Lagrange's method of interpolation, let us say

$$q_1 = f_1(t, c_1, c_2 \cdots), \quad q_2 = f_2(t, c_1, c_2, \cdots), \tag{8}$$

where the c's are the constant coefficients of the various powers of t. We replace these constants by any functions of t which are *entirely arbitrary except that they must reduce to the experimental values* c_1, c_2, *etc., at the instants of observation*. We shall call any such function $A\ (t)$, or simply A. The Eqs. (8) are now replaced by

$$q_1 = f_1(t, A_1, A_2, \cdots), \quad q_2 = f_2(t, A_1, A_2,). \tag{9}$$

In these equations we have a faithful picture of experimental facts. It is necessary, however, to reconsider the concept of mass, since we naturally desire to use it as soon as we consider accelerated motions. In order to obtain a definition of inertial mass suitable for use in the finite difference equations of dynamics, we must describe a procedure by which a number may be assigned to the mass of a body, without any mention of instantaneous velocity, which is a differential coefficient. When we neglect the dependence of mass on velocity, the original definition of Newton is quite satisfactory. That is, if two freely moving bodies collide head on, their masses are inversely as the changes of velocity which they experience. Since the velocity measured is $\Delta x/\Delta t$ rather than dx/dt, we see that the value of a mass, resulting from such an experiment, can be formulated entirely in terms of finite difference coefficients and a constant; namely, the number assigned to the mass of a standard body.

The construction of the Eqs. (8) is especially simple when the time interval between observations is constant. Just as in the illustration of the uniformly moving particle, we are led to finite difference equations. From the standpoint of the principle of real and finite description, these equations, or their integrals (in the sense of the calculus of finite differences), expressed in the form (9), are the true laws of motion.

If, for the moment, we suppose there are no experimental errors, and that the observations are consistent with Newton's laws, we can easily see what the finite difference equations will be. This supposition is made only for pur-

[5] Boole, reference 3, p. 39.

poses of discussion, and implies no recession from the principles laid down in section II. The general difference equation for any problem of particle dynamics, Δt being constant, is obtained by noting that $m\Delta^2x/\Delta t^2$ *is the time mean of the time mean of the force, taken over the interval* Δt. We may verify directly that

$$m\frac{\Delta^2 x}{\Delta t^2} \equiv \frac{1}{\Delta t^2}\int_t^{t+\Delta t}\int_t^{t+\Delta t} m\frac{d^2x}{dt^2}dt\,dt \equiv \frac{1}{\Delta t^2}\int_t^{t+\Delta t}\int_t^{t+\Delta t} F_x dt\,dt,$$

or,

$$m\frac{\Delta^2 x}{\Delta t^2} \equiv \bar{\bar{F}}_x, \tag{10}$$

where the bar denotes a time mean over the interval mentioned. This equation, and two similar ones, constitute a formulation of the laws of motion, to which we should doubtless be led if, in all cases, we were to carry through a process like that which led us to Eqs. (3), (4), and (5). In general the right member of (10) will depend on Δt as well as on t, because the upper limits of the integrals contain Δt. The analogue of (10) in curvilinear coordinates is very complicated. While the writer has found it possible to make considerable progress in working out a theory for second order difference equations analogous to the transformation theory of ordinary dynamics, the results are so involved as to be of little interest. Therefore, we content ourselves with stating Eq. (10).

IV. Relativity Dynamics

In discussing special relativity from the standpoint of the principle of real and finite description, we meet at once with a kinematic problem; namely, the relation between the coordinates in different systems of reference. Whenever we pass to a new system of dynamical variables, the transformation equations themselves should be in accord with the principle. In other words, they may contain difference coefficients, but they should not contain derivatives. It is satisfying to note that the Lorentz transformation may be interpreted in such a way that it obeys this criterion. If the moving system of reference, x', y', z', t', has velocity $V(=Bc)$ parallel to the X-axis of the observer's system, x, y, z, t, this transformation takes the form

$$x' = k(x - Vt),\ y' = y,\ z' = z,\ t' = k\left(t - \frac{Vx}{c^2}\right),\ k = 1/(1 - B^2)^{1/2}. \tag{11}$$

On the usual theory, V represents the *instantaneous* velocity of the moving axes, dX/dt, but for the present purpose a different definition is needed. We may suppose the two sets of axes are attached to isolated bodies between which there is no detectable interaction. If then we find experimentally that $\Delta X/\Delta t$ is a constant V, we *postulate* that the Eqs. (11) hold true, and with this understanding they contain no derivatives. Quite similarly, c represents the quotient of a *measured* distance travelled by a light beam between two stations, and the *measured* time required for its journey. In accordance with the spirit of this paper, c must therefore be expressed as a difference quotient.

If we assume, *momentarily and for purposes of discussion only*, that the relativistic differential equations of motion are valid, the difference equations of motion for a rapidly moving particle may be obtained very easily. The relativistic equations of motion are

$$\frac{d}{dt}(mv_x) = F_x, \text{ etc.} \tag{12}$$

where $m = m_0/(1-\beta^2)^{1/2}$ and $\beta c = v$. If the square of the element of arc in a four-dimensional space is written as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

(12) may be rewritten in the form

$$m_0 c\frac{d^2x}{ds^2} = F_x\frac{dt}{ds}. \tag{13}$$

Suppose now that the observations are uniformly distributed along the world line of the particle, so that we have $\Delta s = \text{constant}$, in place of the former assumption, $\Delta t = \text{constant}$. We proceed just as we did in obtaining (10) with the result

$$m_0 c\frac{\Delta^2 x}{\Delta s^2} = \overline{F_x\frac{dt}{ds}},$$

where the right member denotes

$$\frac{1}{\Delta s^2}\int_s^{s+\Delta s}\int_s^{s+\Delta s} F_x\frac{dt}{ds}\,ds\,ds.$$

There is a way of expressing the law of light propagation without recourse to infinitesimals or to integrals. Since, in the theory of special relativity, the law is $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = 0$, we replace this by

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0. \tag{15}$$

In general relativity also, the law of light propagation may be taken as $\Delta s = 0$.

It is much more difficult to formulate an analogue for the geodesic law $\int ds = 0$, which applies to the free motion of a particle. If we follow the most obvious course and assume that the law should take the form

$$\Delta s = \text{an extremum}, \tag{16}$$

we at once encounter a question of interpretation.

On the one hand, we may assume that we are dealing with a continuous space-time manifold, even though our measurements are discontinuous in character. On this basis Δs means a finite interval between adjoining observations, marked off along a geodesic line, where the word geodesic is taken in its ordinary sense. The significance of (16) is then simply that of a theorem deducible from Einstein's law of motion.

On the other hand, we may suppose that the space-time manifold is simply a discrete aggregate of observed point events, analogous to a two or three dimensional network, like a crystal lattice. The problem of defining a geodesic in such a lattice is not an easy one, as Silberstein has pointed out in his "Theory of Relativity." He suggests that the "shortest distance" between two points a and b in such a lattice (Fig. 2) should be defined as the smallest number of steps which will carry one from a to b. As to the concept of a step, it is necessary to think of the lattice points as joined by a network of lines. A step is then defined as the transition from a given point to its neighbor on one of these lines. In Figs 2a and 2b we have indicated two different methods of drawing such line-networks and it is obvious that the shortest distance depends to a great extent on the nature of these lines. Further, even with such a simple point lattice and network of lines as that shown in Fig. 2a, the paths

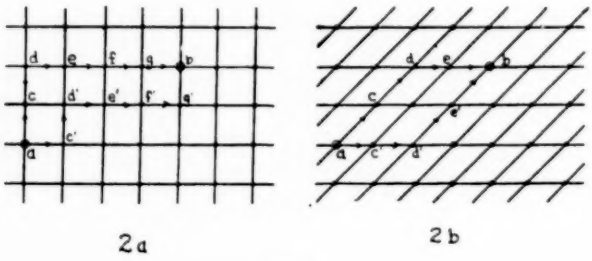

Fig. 2.

which correspond to shortest distances are not unique. Thus, the two paths $a, c, d \cdots b$, and $a, c', d' \cdots b$ both involve the same number of steps, and both may be considered as geodesics between a and b for the particular network shown. Although the shortest distance has a definite value, the path which must be traversed in order to achieve the journey in this shortest distance is not uniquely determined, save in special cases. After all, we should expect that some feature like this would emerge when we deal with discrete quantities.

In view of the existence of a limit to the accuracy with which a coordinate can be measured,[6] we should expect that the concept of shortest distance has an indefinite meaning in the atomic domain. The contribution of the present discussion is to show that a similar difference exists even for macroscopic measurements, as soon as we abandon the idea of a continuous background for the description of events.

V. Discussion of Ehrenfest's Theorem

In sections II and III we emphasized that Newton's laws of motion should be interpreted statistically,—that our knowledge of a single body can never be extensive enough to justify us in supposing it follows the laws of motion exactly. There is a theorem of wave mechanics discovered by Ehrenfest,[7]

[6] Ruark, reference 4.

[7] Ehrenfest, Zeits. f. Physik. **45,** 455 (1927); Ruark and Urey, Atoms Molecules and Quanta, p. 652, McGraw-Hill, 1930.

which throws further light on this situation. Let ψ denote the wave function for a particle, so that $\psi\psi^* dv$ is the probability for the particle to have its coordinates in the element dv of the coordinate space. Then the x-component of the center of gravity of the probability distribution is given by $X = \int \psi^* x \psi dv$, provided the wave function is so normalized that $\int \psi^* \psi dv = 1$, and the theorem may be stated as follows:

$$m\ddot{X} = \int \psi^* \left(-\frac{\partial V}{\partial x} \right) \psi dv. \tag{17}$$

The expressions for $\ddot{Y}$ and $\ddot{Z}$ are similar, and V is the potential energy. Now let us consider a wave packet, having a very small extension compared with any distance in which the potential energy changes appreciably. In evaluating (17), we may use the value of $-\partial V/\partial x$ at the *centroid* of the packet, in every element of the integral, and the result is

$$m\ddot{X} = \left(-\frac{\partial V}{\partial x} \right)_X \int \psi^* \psi dv = \left(-\frac{\partial V}{\partial x} \right)_X. \tag{18}$$

On the basis of an outworn interpretation of wave dynamics, the individual particle is thought of as a wave packet, and $|\psi|$ as the amplitude of the waves. If this were correct, Ehrenfest's theorem would replace the ordinary law of motion, and it would appear foolish to use finite difference equations in describing the motion of a particle. But, on the probability interpretation, the meaning of Ehrenfest's theorem is this: We consider a great number of similar experiments, in each of which a single particle moves along the x-axis, and if we measure the x-coordinate of each particle at a definite instant, we obtain the distribution of $\psi\psi^*$ appropriate to that instant. At any future time the distribution can be calculated from the wave equations and we have a picture of the *statistical* behavior of the particles at that time. In other words, the wave equations enable us only to follow the time changes of the probability distribution, and there is room for uncertainty in the motion of the individual particles. Only the centroid obeys the approximate law of motion (18). Now, from a knowledge of the probability distribution obtained by a great number of experiments on individual isolated particles, we can draw no definite conclusions as to the motion of the centroid of a great number of particles, bound together in a rigid body and set in motion all together. This remark should make it clear that there is nothing in Ehrenfest's theorem which contradicts the conclusions of sections II and III.

VI. Difference Equations in Electrodynamics

The use of difference equations in electromagnetic theory arises naturally from our discussion of mechanics. Details will have to be reserved for a later occasion, but the leading ideas will be indicated here. To begin with, the electromagnetic field is *defined* entirely in terms of dynamical measurements. For our present purpose, the forces between charged bodies, magnets, and conductors carrying currents, are the fundamental things. We may employ

a small test body carrying a charge e, to map out a field of electric force. Again, if the test body is endowed with a "velocity" $\Delta s/\Delta t$, chosen to suit our convenience, a field of magnetic force may be studied. Thus we obtain the distribution of six quantities, E_x, E_y, E_z, H_x, H_y, and H_z, which might be called components of electric and magnetic intensity if the names were not preëmpted by analogous entities incapable of being measured, and defined in terms of the behavior of infinitesimal test bodies. The quantities $E_x, \cdots H_z$ are defined *by the very processes used in measuring them*. There is no implication that they obey Maxwell's equations. Indeed, they do not, because, from the usual standpoint, they represent average values of the field intensities, measured by a finite test body and not by a mathematical abstraction. These quantities may well be referred to as the *physical* field intensities. Their properties are simply the properties of numbers which fit a modified Lorentz equation,

$$e\boldsymbol{F} \equiv e\left(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}\boldsymbol{H}]\right). \tag{19}$$

In this equation v means $\Delta s/\Delta t$ rather than ds/dt; e is measured by experiments whose interpretation requires no use of infinitesimals and eF_x is a force measured, let us say, by readings on the scale of an actual spring balance, or by equating it to the value of $m\Delta^2 x/\Delta t^2$ for the test body. The field equations obeyed by the physical field intensities may now, in principle, be found by trial. They involve the properties of the test body as well as those of the bodies producing the field—a point which is important in itself. The same consideration applies in mechanics but was not emphasized in section III, for the sake of simplicity.

VII. Conclusion

In conclusion, this paper serves to introduce a method of talking consistently about the results of experiments, and attention may be directed again to the refinements hinted at in the first foot note. The writer's aim has been to show how physical facts can be formulated by the use of difference calculus, that is, essentially with the aid of rational numbers. Considering the almost universal use of idealized concepts such as infinitesimals and irrationals, it seemed worth while to examine the possibilities of proceeding otherwise, and the results may be summarized in a broad sense,as follows: a method has been outlined by which we can describe experimental results mathematically, without any implications concerning the state of a system at times when it is not being observed, or concerning the causal connections customarily implied by the use of continuous analysis.

The writer wishes to thank Doctors Hutchisson, Muskat, Rashevsky and Worthing for helpful discussions.

LETTERS TO THE EDITOR

Prompt publication of brief reports of important discoveries in physics may be secured by addressing them to this department. Closing dates for this department are, for the first issue of the month, the twenty-eighth of the preceding month; for the second issue, the thirteenth of the month. The Board of Editors does not hold itself responsible for the opinions expressed by the correspondents.

Nuclear Spin

On the basis of measurements by Rasetti[1] of the Raman effect in N_2, Heitler and Herzberg[2] have suggested that the electron in the nucleus has lost not only its spin, but also its power to determine what the statistics of the nucleus shall be. (One could say, perhaps, that the electrons in the nucleus behave as if they had no spin and obey, therefore, the Bose-Einstein statistics, just as photons do.) The purpose of the present note is to point out that *the assumption that the protons alone contribute to the resultant spin is sufficient to explain all the known facts.* Furthermore, it may be that there is a building-up principle analogous to that of the outer electrons, although the evidence[3,4] is insufficient to show exactly what this would be.

TABLE I. *Building-up process.*

Element	P	Z	I	Protons added	Spins added
H	1	1	1/2		
He	4	2	0	3	$(+-)-$
Li	7	3	3/2	3	$+++$
C	12	6	0	5	$---(+-)$
N	14	7	1	2	$++$
O	16	8	0	2	$--$
F	19	9	1/2	3	$(+-)+$

The table gives in the last two columns the number of protons and spins, respectively, which must be added to the element in the row above to give the element in the row itself.

Thus, if we suppose that each additional proton contributes a spin of $\pm\frac{1}{2}$ to the resultant (which is the simplest assumption), the elements may be ordered as shown. Starting with He^4, $I=0$, three parallel spins will produce Li^7, $I=3/2$, and five more, two forming a pair, can result in C^{12}, $I=0$. Two additional parallel spins give N^{14}, $I=1$; two more, opposite in direction to the first two, O^{16}, $I=0$; and three more, two forming a pair, can yield $I=\frac{1}{2}$ for F^{19}. The value of nuclear spin for Na^{23} is in doubt, but if we assume $I=5/2$, then, by interpolation, Ne^{20} should have $I=1$ and Ne^{22}, $I=2$. One would also expect, since $I=0$ for C^{12} and S^{32}, a value of $I=1/2$ for B^{11} and P^{31}.

The difficulty mentioned by Goudsmit for the cadmium spectrum also disappears, if one adopts the above hypotheses. The value of $I=1/2$ may be attributed to Cd^{111} or Cd^{113}, but there exist no criteria for choosing between the two isotopes. (Perhaps both give a hyperfine structure.)

It seems, therefore, that the experimental evidence to date indicates the validity of the rule that even numbers of protons imply integral values for I, and odd numbers half-integral values. This is true for Mn, Br, Pr and Bi.

One would expect aluminium, phosphorus, chlorine, potassium, scandium, vanadium, cobalt, copper, gallium, arsenic, and other elements with isotopes having only odd numbers of protons to have half-integral values for the nuclear spin.

It remains an open question as to whether or not the orbital angular momentum of the protons plays a rôle, and also whether or not closed shells of protons exist, as do closed shells of extra-nuclear electrons. If there are such closed shells of protons, then this might provide the explanation of inverted hyperfine structure, the inversion being due to the absence of a proton from such a shell.

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January 15, 1931.

[1] F. Rasetti, Proc. Nat. Acad. Sci. **15**, 515 (1929).

[2] Heitler and Herzberg, Naturwiss. **17**, 673 (1929).

[3] R. S. Mulliken, Trans. Faraday Soc. **25**, 634 (1929).

[4] Pauling and Goudsmit, Line Spectra (1930).

The Cathode-Ray Tube in X-Ray Spectroscopy and Quantitative Analysis

Under this title the Journal of the American Chemical Society has published a paper by us in the January number. A Coolidge tube was used, having a soldered aluminum window 6 mm in diameter and operated continuously at 0.2 m.a. and 85 kv D.C. Experiments were made with metal targets placed above the window and cooled with a jet of hydrogen. For the recording of the relative intensities of the $K\alpha$ doublet, primary attention was given to a method involving the application of an ionization chamber and an amplifying set. The latter made use of the new FP54 tube (Metcalf and Thompson, Phys. Rev. **36,** 1489 (1930)). The smallest measurable grid current was 8.7 $(10)^{-16}$ amps. With a copper target and 0.4 mm slits, the peak of the $K\alpha$ doublet gave a net deflection of 90 cm with an accuracy of 0.5 cm on a scale 2.6 m distant, corresponding to an ionization current of 3 $(10)^{-13}$ amp.

The loss in electron velocity in passing through the window agreed with the Thomson formula. For molybdenum and iron the intensity was found to vary as the square of the difference between the electron velocity and the exciting voltage. This relationship held up to the maximum velocity employed, amounting to four-fold the exciting voltage for the case of molybdenum and eleven-fold for iron. Experiments as yet unreported are being made on titanium and calcium and the exponent has been found to be 1.3 for the voltage range extending from ten to sixteen-fold the exciting voltage. The exponent accordingly decreases less rapidly with increase in voltage than has been observed in similar experiments with x-ray tubes.

Curves showing the relation between chemical composition and the relative intensities of the unresolved $K\alpha$ doublets were obtained for copper-nickel and iron-nickel alloys—the latter representing the case in which the intensity of radiation from one constituent is increased as the result of excitation by the other.

Other experiments were carried out, using photographic recording by means of a Seemann spectrometer. For a molybdenum target, a 1° oscillation of the crystal and a slit width of 0.05 mm, a density of the $K\alpha_1$ line amounting to 0.4 was obtained by an exposure of sixteen minutes. Analyses by this method were made on silver-cadmium and tin-antimony alloys and the results were confirmed by chemical examination.

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The Hyperfine Structure of Ionized Lithium

The hyperfine structure of the low ^3P state of Li^+ is of the same order of magnitude as the triplet separation itself. One might expect large deviations from the interval rule for hyperfine splitting, corresponding to large contributions of one triplet level to the perturbed eigenfunctions of another. Since the Li^+ hyperfine structure has not yet been explained conclusively,[1] we have estimated the effect of these deviations on the calculated positions of the components. The resulting corrections are, however, small, and are not sufficient to give a better agreement than obtained by Güttinger.

Our method is simply to consider the transition between the extreme cases wherein the hyperfine splitting is either small or large compared to the triplet separation. The coefficients of general equations for the levels were determined by use of the vector model in the extreme cases, and by using the usual identification of the lines for the isotope Li_6. The nucleus was considered as coupled only to the $1s$ electron in the ^3P. About 6 percent was subtracted from the coupling constant as determined from the ^3S, due to the interaction with the $2s$ electron there.[2] This correction alone has some effect on the results of Güttinger, as is indicated in the third column of the table.

In the case of $i=1\frac{1}{2}$, one of the coefficients of a cubic equation cannot be determined, and is represented by x in Table I. Its order of magnitude should be one-tenth cm^{-1}. Its evaluation would be intricate, but no value

[1] H. Schüler, Zeits. f. Physik **66,** 431 (1930). P. Güttinger, Zeits. f. Physik **64,** 749 (1930).

[2] G. Breit and F. Doermann, Phys. Rev. **36,** 1732 (1930).

TABLE I.

			$i=1\frac{1}{2}$			$i=\frac{1}{2}$
Line[3]	Schüler (exp.)	Güttinger	Extreme coupling	Intermed. coupling		Intermed. coupling
1	(0.5) 18231.31	(1) 31.31	31.31	31.31		
2	(1) 30.91	(2) 30.91	30.91	30.91	(1)	30.91
3	(2) 30.26	(3) 30.25	30.26	30.26	(2)	30.26
4c		(2) 27.82	27.79	27.83−x		
4b	(6) 27.72	(5.5) 27.74	27.88	27.97−3.1x	(5)	27.56
4a		(2) 27.62	27.69	27.70−x		
5b	(12) 27.52	(11) 27.56	27.58	27.58−x	(9)	27.39
5a		(3) 27.42	27.48	27.57−3.1x		
6c		(.5) 27.22	27.29	27.30−x		
6b	(2) 27.15	(2.5) 27.10	27.14	27.18−x	(1)	26.91
8c		(2) 25.96	26.03	26.12		
8b	(6) 25.91	(2.5) 25.89	25.94	25.91−x	(1)	25.80
13b		(.7) 25.56	25.63	25.72		
13a	(1.5) 25.46	(2) 25.36	25.44	25.44−x	(2)	25.51
9	(4) 25.25	(5.5) 25.23	25.29	25.26−x	(5)	25.15
10	(1) 24.93	(2.5) 24.90	24.98	25.07	(1)	24.86

would bring much better agreement than was found in Güttinger's approximation. In the case $i=\frac{1}{2}$, our results give agreement just slightly worse than those quoted by Schüler.

[3] The components 7, 11 and 14 are attributed to the Li^6 isotope and have been omitted in this table. They are at 26.43, 24.33 and 29.48.

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January 19, 1930.

An Attempt to Detect Axiality of X-Ray Emission

J. Stark[1] has recently reported that fluorescent K-series x-rays are not emitted with equal intensity in all directions from certain crystals, and that these x-rays are at least partially plane-polarized. In the case of γ-γ-dibromoanthracene, a monoclinic crystal of needle-like habit with the b-axis parallel to the length of the needle, he found about 20 percent more bromine K-radiation in directions 90° from the b-axis than in a direction 10° therefrom. The scattering of this radiation by aluminium indicated that it was polarized with the electric vector parallel to the b-axis. These effects are so important from the theoretical side that independent confirmation seemed desirable. This note reports an unsuccessful attempt to detect differences in the intensity of zinc K-radiation emitted in different crystallographic directions from a zinc crystal.

Zinc was chosen because the axial ratio for its hexagonal crystals, $c/a=1.86$, is so much greater than that (1.63) to be expected in a close-packed arrangement of spheres that it is reasonable to suppose the atomic axes are not free to rotate about axes in the (001) plane. This is one of the criteria suggested by Stark for selecting materials likely to show the new effect. His objection to the use of metals, that the atoms therein are presumably free to rotate, does not therefore seem valid with respect to zinc.

The primary x-rays used were from a tungsten target tube operated at 55000 volts (peak) or from a molybdenum target tube operated at 44000 volts (peak). The zinc monocrystalline plate, 0.4 cm thick was rotated about its normal into various azimuths while its secondary radiation in a fixed direction 90° from the primary beam was measured by balancing its ionization with that of part of the primary beam. The absorption coefficient of the secondary radiation was found to be that of zinc K-radiation, indicating that the scattered radiation at 90° was negligible in comparison with the fluorescent. The normal to the plate was found by Laue and rotation

[1] Stark, Ann. d. Physik [5] **6**, 637–662 (1930).

photographs to lie at 20° from the hexagonal axis of the crystal. so that if θ is the angle between the normal and the axis of the principal ionization chamber the angles between this axis and the hexagonal axis of the crystal vary between $\theta+20°$ and $\theta-20°$ during the rotation of the specimen. For the three settings tested, θ was 52°, 62° and 77°. In no case was the variation in observed intensity as much as one percent. We conclude that at least as regards intensity the effect reported by Stark is unimportant in zinc.

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January 23, 1931.

Rotational Fine Structure in Raman Spectra

In recording the rotation spectrum of a molecule by scattered light one of the major difficulties is the very great intensity of the unmodified line as compared with those lines arising from changes in rotational energy. Long exposures tend to broaden the strong line and obliterate the weaker ones which lie very close to it. An ingenious arrangement employed by Rasetti (Zeits. f. Physik **66,** 646 (1930)) is partially effective in overcoming this handicap. He introduced mercury vapor into the spectrograph to absorb from the scattered light those unmodified frequencies which arise from the normal state of the atom. The observations are thus practically confined to λ2537 as parent line. Even this expedient is not wholly successful, since λ2535 is superposed upon the rotation pattern and appears very intense in comparison with it.

Recently Manneback (Zeits. f. Physik **62,** 224 (1930)) has computed the relative intensities and polarizations for Raleigh and Raman scattering of lines with and without rotational displacements. He shows that if the incident light is plane polarized and the scattered light is observed in a plane normal to the electric vector, the rotation lines are almost completely depolarized. Since the unmodified line is not depolarized, this suggests the introduction of a Nicol prism between the camera and the scattering chamber. Any convenient source and any frequency for the incident line could then be used, and not only the parent line but also neighboring lines scattered without change in frequency would be eliminated. In all probability the background illumination would also be materially lessened. With this improvement in contrast considerably smaller total exposures should prove adequate, and the necessary time might not be much extended, even though the illumination would be considerably fainter.

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January 28, 1931.

Biological Effects of Gamma-Rays

In connection with the development ir various laboratories of x-ray tubes operating at voltages of the order of a million volts, the question of adequate protection of those engaged in the work against the very penetrating gamma-rays emitted by such tubes has be come of considerable importance. It has seemed valuable to perform some experiments with highly filtered gamma-rays from radium in order to ascertain the dangers from exposure to the penetrating radiation, which may be considerable in amount, passing through whatever shielding may be used around the tube. Estimates doubtless might be made from previous work, but for comparison it appeared desirable to have data on exposures to gamma-rays from radium from which all soft components were similarly filtered out.

A group of 63 white rats (with known antecedents) has been exposed to the radiation from 6 grams of radium filtered through one mm of platinum, one mm of brass, 16 mm of lead, and 5 mm of celluloid at a distance of 41 mm to the nearest side of the rat. For four additional rats the celluloid was omitted. The rats were put in celluloid boxes 47 mm high placed above and below the radium box. The radium was spread out over a surface of 3 by 10 cm area to give more uniform exposure. With this large source-area and distance, the exposure was reasonably similar throughout the whole body of the rat. In addition, 16 rats were exposed to 2.5 grams of radium at the same distance but with the lead filter removed; for eight of this group the celluloid was also omitted. This group of 16 thus re-

ceived approximately the same exposure to the hard components of the radiation but with the addition of that amount of soft gamma-radiation which penetrated the platinum and brass. The celluloid was used in order to reduce the beta-ray intensity to about the level corresponding to the rate of beta-ray production in the tissues. Exposure-times were varied from one-half minute to 17 hours in a roughly geometric progression. The 2.5 grams without the lead filter should give an "erythema dose" in about four hours at the average distance of 6 cm according to ordinary clinical data.

All rats died which were exposed six hours and longer to the 6 grams of radium with filters. With a 16-hour exposure, death ensued in three or four days, due at least in part to injury to the gastro-intestinal tract. With six-hour exposures the rats showed similar but less severe gastro-intestinal symptoms from which all recovered in about three days but died one to three weeks later from a progressive anemia affecting both red and white blood-cells. One of the six rats exposed to three hours died from a similar anemia after 11 days. The lethal exposure lies thus between three and six hours, or about four hours. Two of four rats exposed 2.5 hours to 2.5 grams of radium without the lead filter died from anemia after 19 and 37 days. The lethal exposure for these conditions is thus about three hours.

Blood-counts were made on 66 rats before and 0, 1, 3, 7, 14, 21, and 35 days after exposure. The minimum exposure which showed an appreciable effect on the blood-count was approximately 20 minutes for the rats exposed to the 6 grams with filters. Since a single massive dose is the most favorable for detecting a change in blood-count and since ten times this exposure is above the lethal dose, blood-count observations alone are not a satisfactory protective measure for those working with high-voltage tubes. Breeding experiments are in progress, and the evidence so far indicates that the sterility dose approximates the lethal dose under these conditions. Abnormalities have to date appeared only in the litter of one female fertilized four weeks after a three-hour exposure.

To provide a suitable index of the total exposure of investigators working with high-voltage tubes, a series of Eastman extra fast (CC) dental x-ray films was enclosed in a lead cassette having walls 6.9 mm thick, and this cassette was exposed in the mid-position of the rat (6.2 cm to the film) for periods of 10^{-5} to 10^{-1} lethal exposure-time as determined for the rats. An exposure of 1×10^{-3} lethal dose gives a point just at the lower end of the straight portion of the density-log exposure-curve, that is, 7×10^{-4} lethal dose gives a density which is easily confused with fog due to development, whereas 3×10^{-3} gives a very obvious blackening of the film. Densities as measured on a photoelectric photometer for the two films in each placque are given below. The placque was exposed as labeled with the "black paper toward tube" (radium).

Fraction of lethal exposure-time	Percent transmission incident film	Percent transmission emergent film	Average T	Density = $\log_{10} 1/T$
None	100	100	1.00	0.0
0.00005	100	100	1.00	0.0
0.0001	83	97	0.90	0.045
0.0003	65	83	0.74	0.13
0.0007	46	64	0.55	0.26
0.001	33	45	0.39	0.41
0.003	9.5	12	0.11	0.96
0.007	1.8	1.8	0.018	1.74
0.01	0.71	0.82	0.0075	2.12
0.05	0.028	0.033	0.0003	3.5
0.10	0.028	0.013	0.0002	3.7

For these film-exposures 2.5 grams of radium were used with the 16-mm lead filter (in addition to the cassette), and 12 hours was taken as the lethal exposure-time.

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BOOK REVIEWS

Vorlesungen uber Wellenmechanik. A. Landé. Pp. 132, figs. 15. Akademische Verlagsgesellschaft, Leipzig 1930. Price RM 9.50.

This small volume represents the lectures delivered at Ohio State University during the academic year 1929–30. In general character it is more of a Handbuch than a self-sufficient introductory treatise. The number of topics touched upon is surprisingly large, but the treatment of the individual items is correspondingly shortened.

After a brief introduction on the general foundations of atomic theory there is a section on the dualism of the wave and corpuscle viewpoints, and one on the uncertainty principle. The treatment is along the usual lines, but the compactness of the material is so great as to detract very considerably from its value. The third section on quantum statistics gives a treatment of the Bose-Einstein method but only a mention of that of Fermi.

The major portion of the book is of course devoted to Schrödinger's equation and its solution. Only the simplest cases are discussed, but no attempt is made to give actual methods of solution, results only being stated. One may call attention to the treatment of the Pauli principle and the symmetry properties of wave functions, which is rather more carefully stated than is usual. The last section on the relativistic wave equation contains a paragraph on Dirac's electron theory and even one on quantum electrodynamics. Perhaps one of the best uses of this book will be as a guide in the development of more detailed lecture notes.

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Handbuch der Astrophysik. Band II, Erste Hälfte. Edited by G. Eberhard, A. Kohlschutter and H. Ludendorff. Pp. 430, figs. 51. Julius Springer, Berlin, 1929. Price bound RM 69.

The importance of photometry in astrophysical work is well illustrated by the fact that this volume of 430 pages forms but a half of the section of the *Handbuch* which is to be devoted to this subject. The first and longest chapter deals with photometric theory. The photometry of the stars, and of self-luminous bodies in general, is rather summarily treated, for example, nothing is said about the measurement of the brightness of the sun relative to the stars, or of the comparison between the astronomical standard of light (expressed in stellar magnitudes) with the terrestrial standard (meter-candles). On the other hand, the photometry of diffusely reflecting bodies—the moon, the planets, Saturn's rings, the zodiacal light, etc., and the extinction of light in the earth's atmosphere are discussed in great detail, with the inclusion of important investigations by the author (Professor E. Schoenberg) some of which are new, while others have been difficult of access. This complete monograph, accompanied by extensive tables should be of much value to all workers in this field. There are few references, however, to work of later date than 1922, and it is to be regretted that no mention is made of Danjon's work on the earth-shine upon the moon, or of Hubble's on the illumination of nebulae by neighboring stars.

The chapter on spectro-photometry (by Dr. A. Brill) deals with a subject so new that it is largely an account of the methods employed by individual investigators. These are clearly presented and the summary is brought up to the end of 1927—including some of the modern work on the contours of individual spectral lines.

A short chapter on colorimetry (by K. Bottlinger) contains an admirable survey of the meaning of color-indices and related magnitudes, and a critical discussion of the methods and results of observation up to 1926. The final chapter on photoelectric photometry, (by H. Rosenberg) contains a thorough discussion of photoelectric and selenium cells. The latter have been practically abandoned in favor of the former, which are of ever-increasing importance. He justly remarks that their introduction into observatories has doubtless been delayed by the unfamiliarity of astronomers with the details of delicate electrical equipment, and his full descriptions of instrumental design and wiring should be of real aid here.

Henry Norris Russell